Menhirs in the desert regions of a Poisson Point Process

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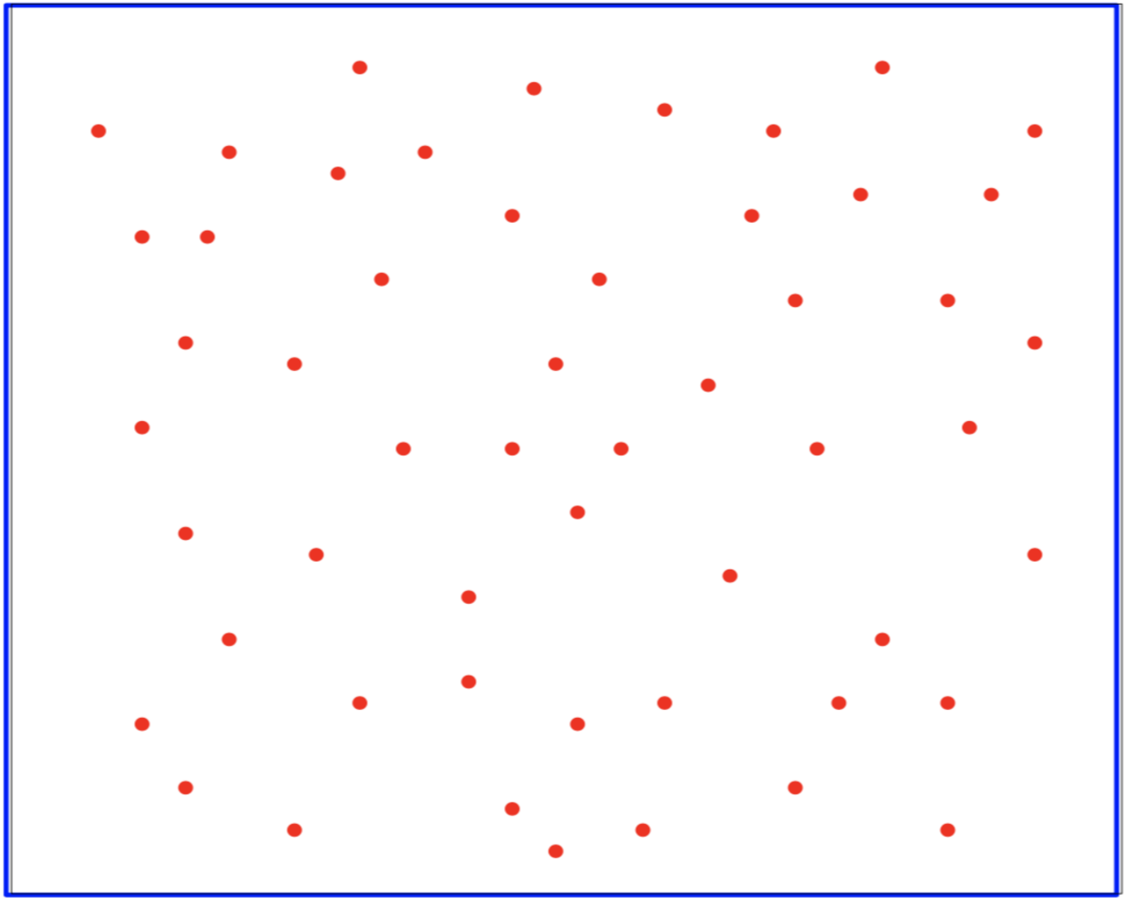
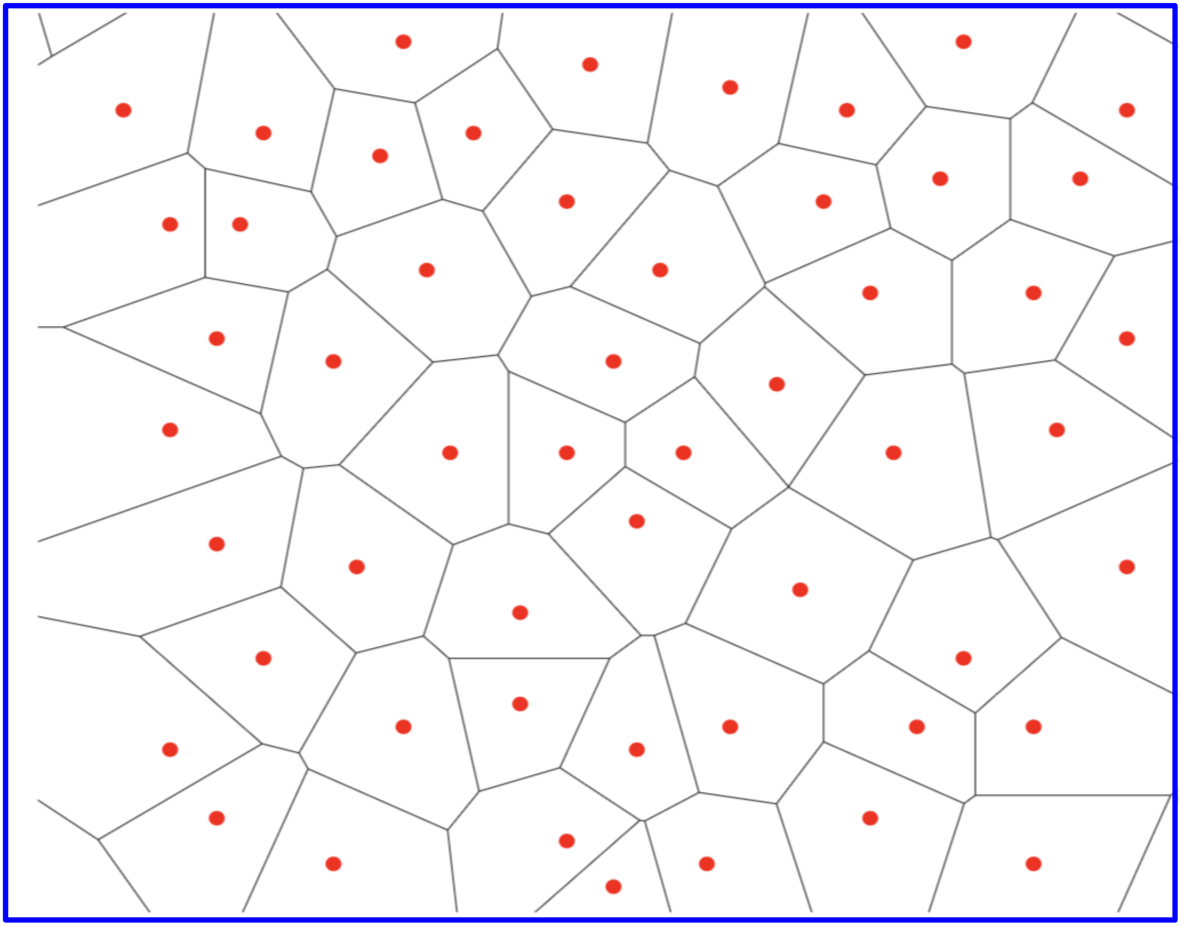
Joint work with **Pierre Calka** and **Yann Demichel**

Let us consider a **uniform Poisson Point Process** in the plane

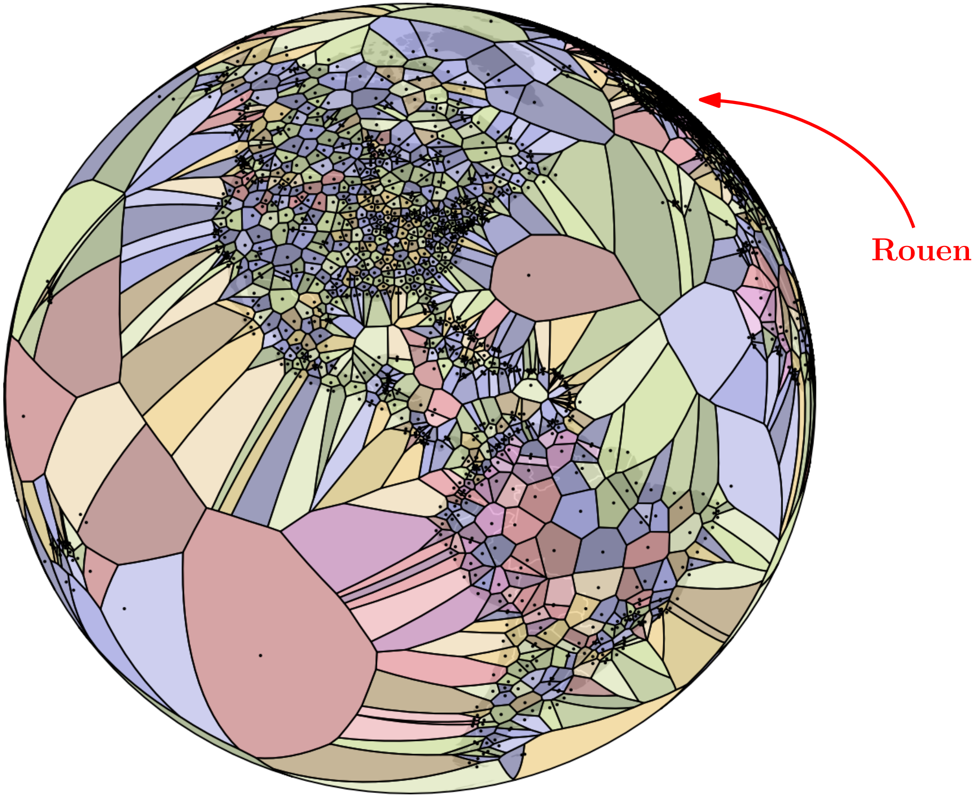


The iconic result in this subject says that the **mean number of vertices** of a cell is **equal to 6**

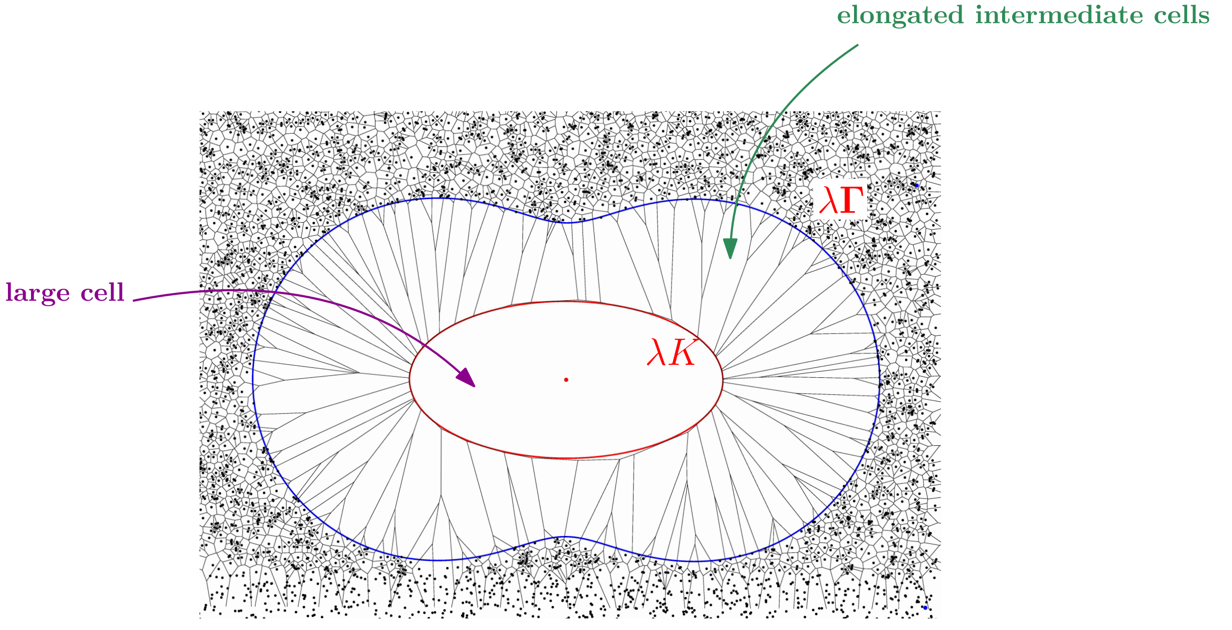


1 - When the underlying PPP has a uniform intensity outside a large region

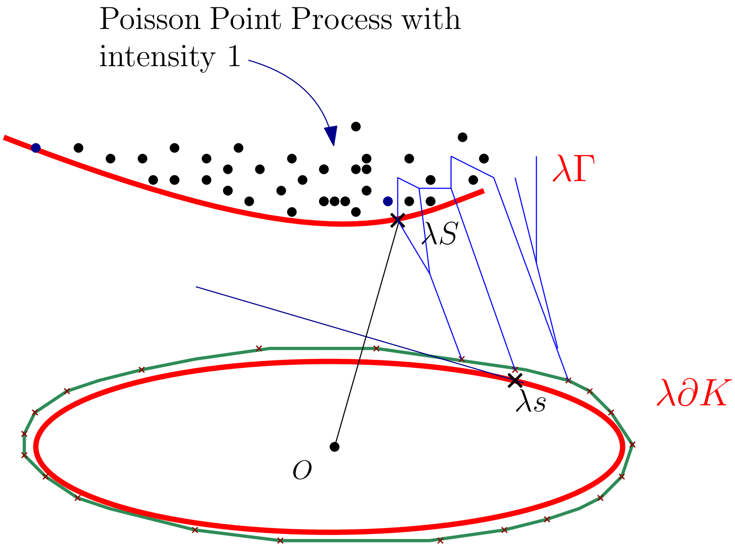
An example: the map of the airports over the world



A simulation with an isolated point : two types of cell



1. The number of vertices of the large cell

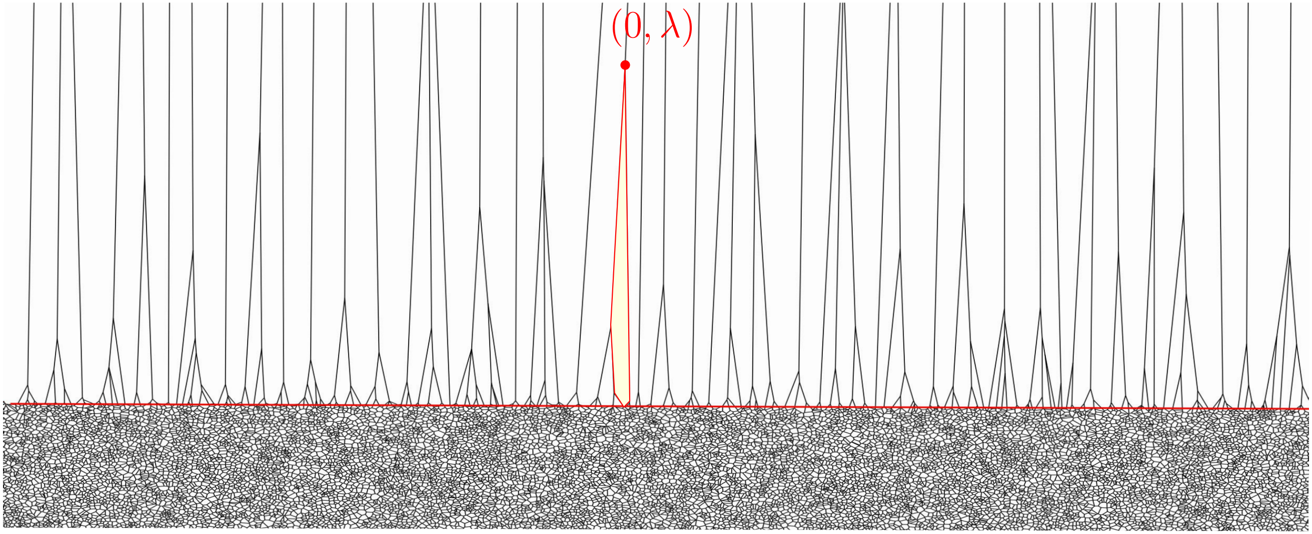


Theorem. (P. Calka, Y. Demichel, E. 2021)
When λ goes to ∞, the intensity of the projection of the vertices on ∂K is asymptotically

$$C.\lambda^{-1/3}.\frac{dist(O,S)^{1/3}}{r_s^{2/3}}ds$$

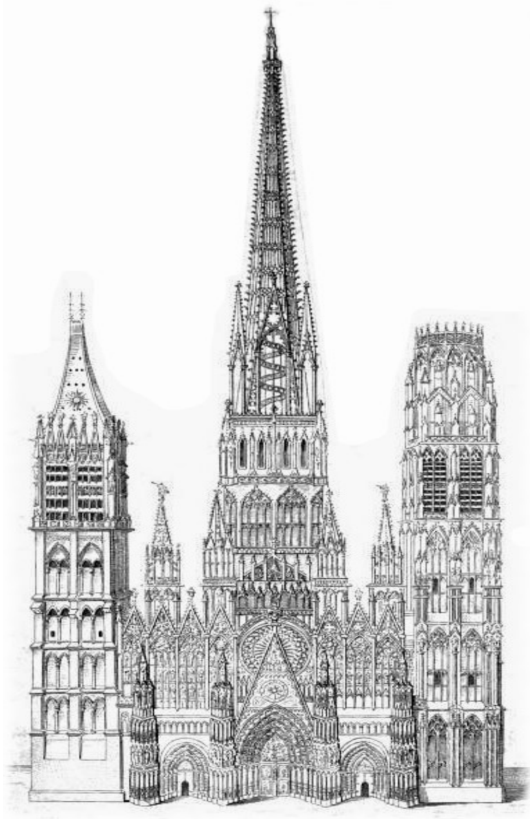
where r_s is the curvature of ∂K at point s, and ds is the Lebesgue measure on ∂K.

2. The Voronoi diagram of a uniform PPP outside the upper half-plane



Goal: Study the asymptotic properties of a cell conditioned to have its highest point at $(0, \lambda)$, when $\lambda \to \infty$.

Some familiar picture?



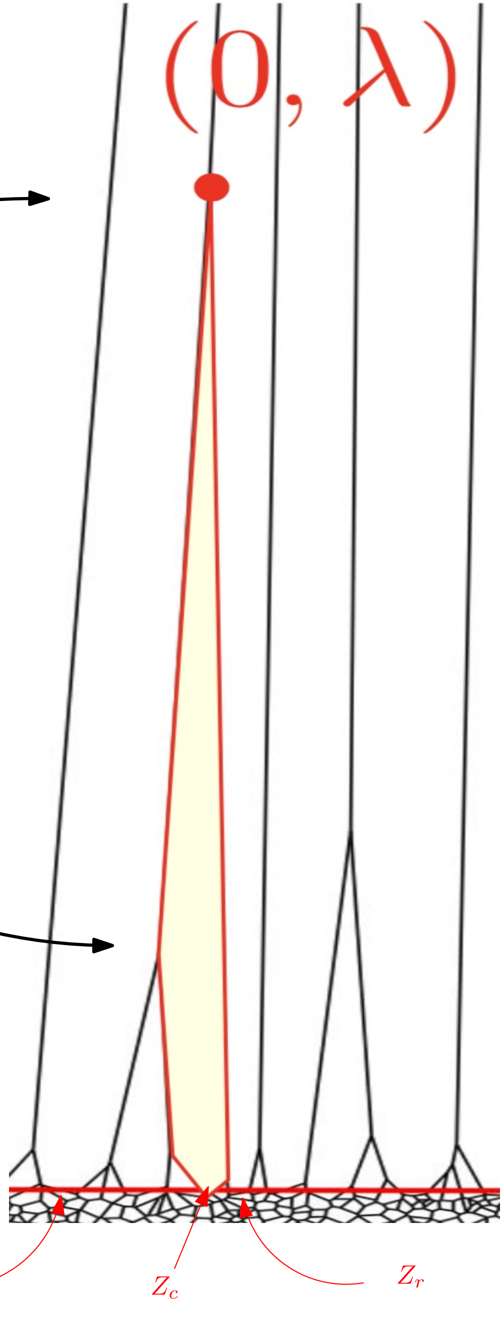
Goal: **Study the asymptotic properties of cathedrals...**

Zooming on a cell

Two steps:

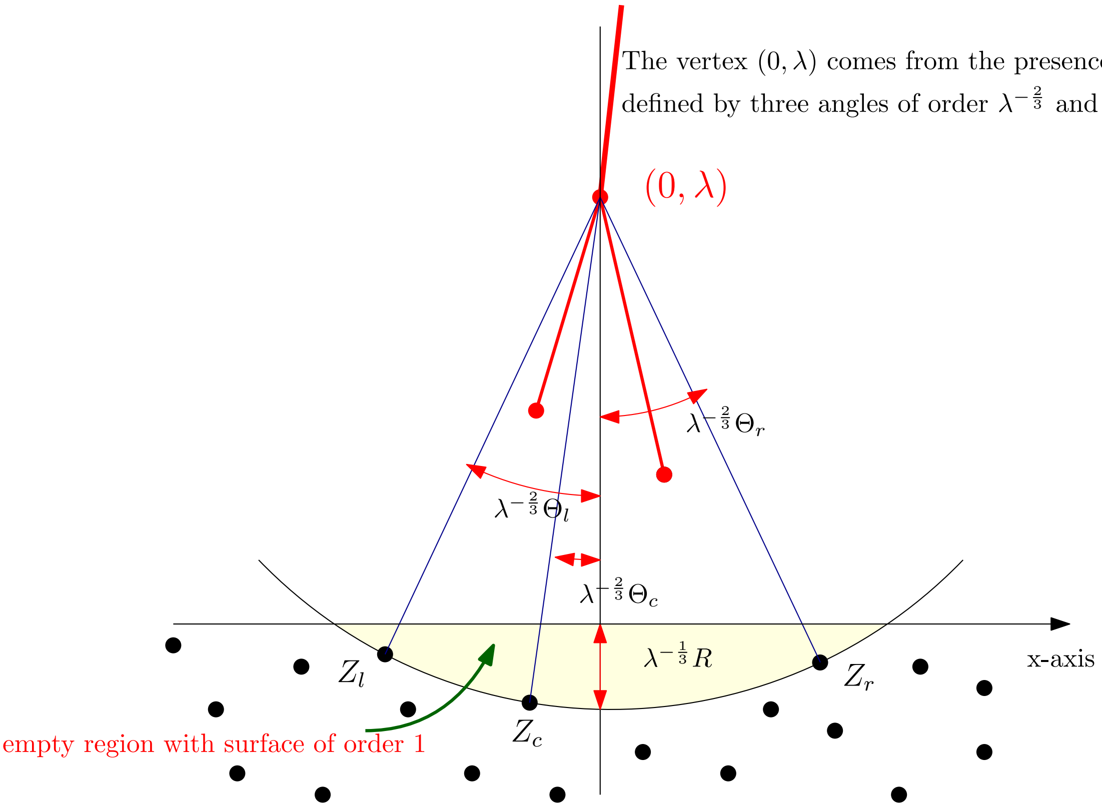
Step 1: the starting point

Step 2: the evolution of the boundary



2a. The 3 bisecting lines starting from the point $(0, \lambda)$, conditioned to be a vertex

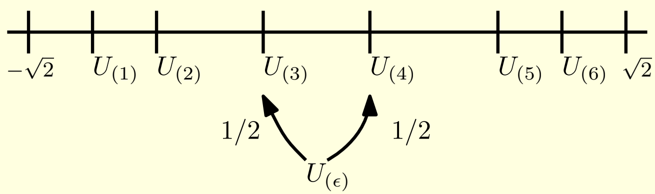
The vertex $(0, \lambda)$ comes from the presence of three germs Z_l, Z_c and Z_r defined by three angles of order $\lambda^{-\frac{2}{3}}$ and an extra radius of order $\lambda^{-\frac{1}{3}}$.



Proposition. As λ goes to infinity, the quadruplet $(R, \Theta_l, \Theta_c, \Theta_r)$ converges in distribution to a distribution proportional to

$$\exp\left(-\frac{4\sqrt{2}}{3} r^{\frac{3}{2}}\right)(\theta_c - \theta_l)(\theta_r - \theta_c)(\theta_r - \theta_l)\mathbf{1}_{-\sqrt{2r} < \theta_l < \theta_c < \theta_r < \sqrt{2r}}.$$

In other words, the limiting quadruplet $(R, \Theta_l, \Theta_c, \Theta_r) \sim (G^{\frac{2}{3}}, G^{\frac{1}{3}} U_{(1)}, G^{\frac{1}{3}} U_{(\epsilon)}, G^{\frac{1}{3}} U_{(6)})$ where G is Gamma distributed and



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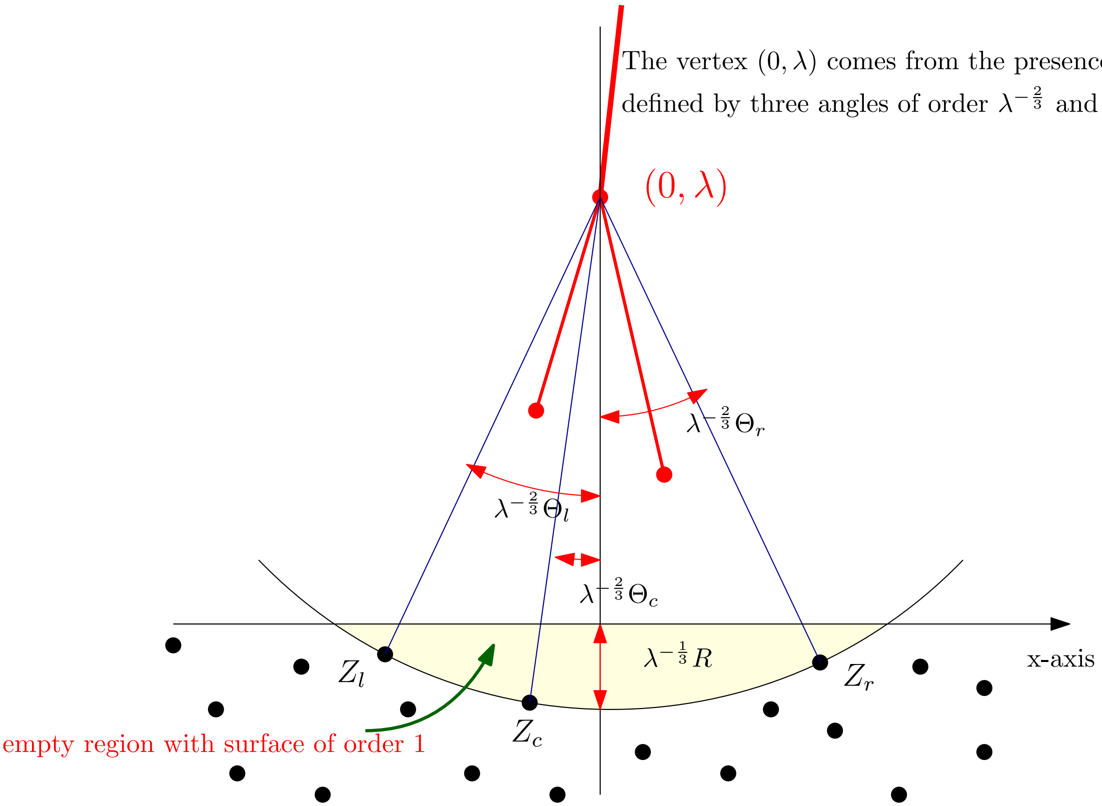

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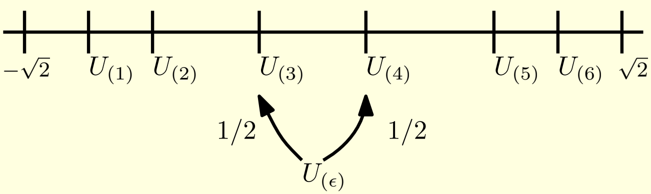


Proof: Applying **Mecke's principle**, we conduct the following computation:

$$E\Big[\sum_{v\in V(\mathcal{P})} f(v,\mathcal{P})\Big] = \frac{1}{3!}E\Big[\sum_{(x,y,z)\in\mathcal{P}^3} f(c_{x,y,z},\mathcal{P}\setminus B_{x,y,z})\mathbf{1}_{V(\mathcal{P})}(c_{x,y,z})\Big]$$

$$= \frac{1}{6}\int_{(\mathbf{H}^c)^3} E\big[f(c_{x,y,z},\mathcal{P}_{x,y,z}\setminus B_{x,y,z})\mathbf{1}_{V(\mathcal{P}_{x,y,z})}(c_{x,y,z})\big]dxdydz$$

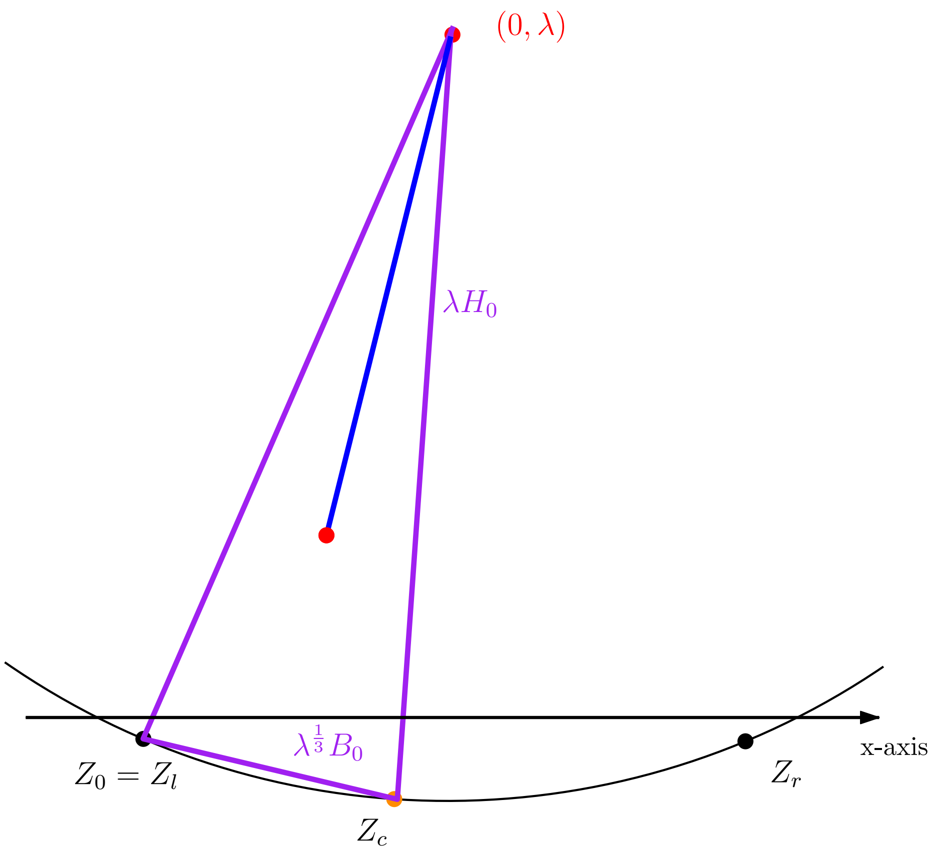
$$= \int_{\{x_1<y_1<z_1\}} E\big[f(c_{x,y,z},\mathcal{P}_{x,y,z}\setminus B_{x,y,z})\big]\; e^{-|B_{x,y,z}\cap\mathbf{H}^c|}dxdydz$$

where $c_{x,y,z}$ denotes the center of the circumscribed ball $B_{x,y,z}$ to the points x,y,z and $\mathcal{P}_{x,y,z}$ denotes $\mathcal{P}\cup\{x,y,z\}$.

The second step is also classical and consists in applying the **Blaschke-Petkantschin formula**, which is nothing but a change of variables where the triplet $\{x,y,z\}$ is parametrized by $c_{x,y,z}$, the radius of the ball $B_{x,y,z}$, and three angles on this ball.

The Jacobian in this change of variables is proportional to a product of three sinus of difference of angles which gives rise after taking the limit $\lambda\to\infty$, to the term $(\theta_c-\theta_l)(\theta_r-\theta_c)(\theta_r-\theta_l)$.

2b. Following a cell down to the boundary



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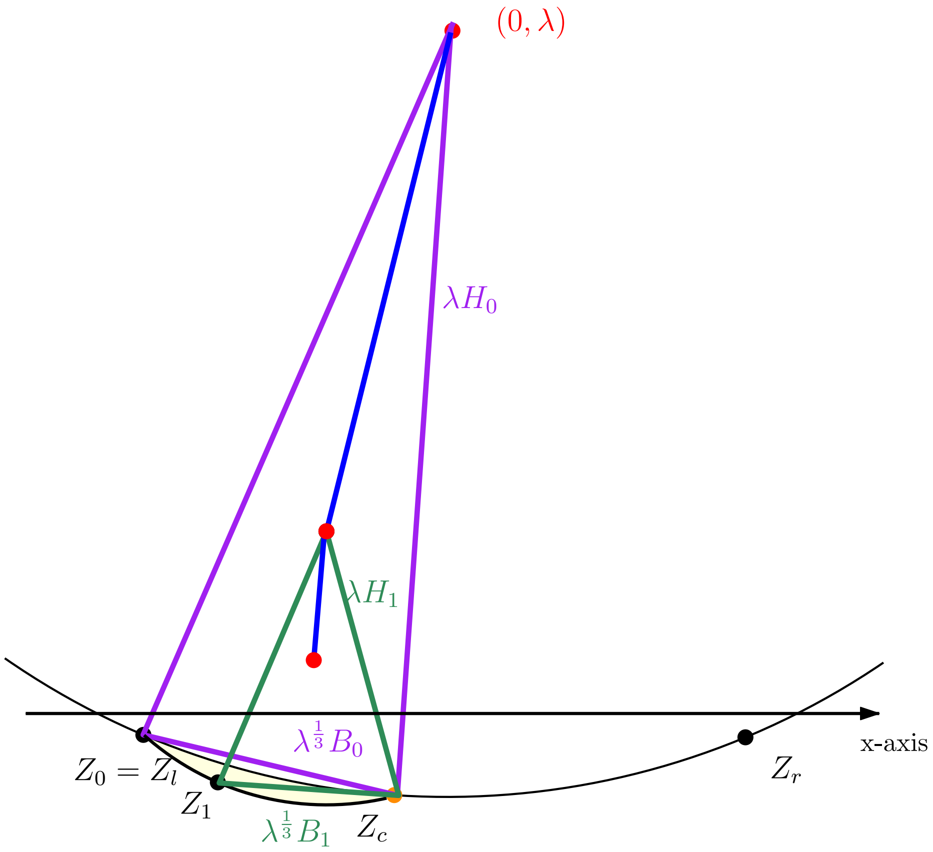

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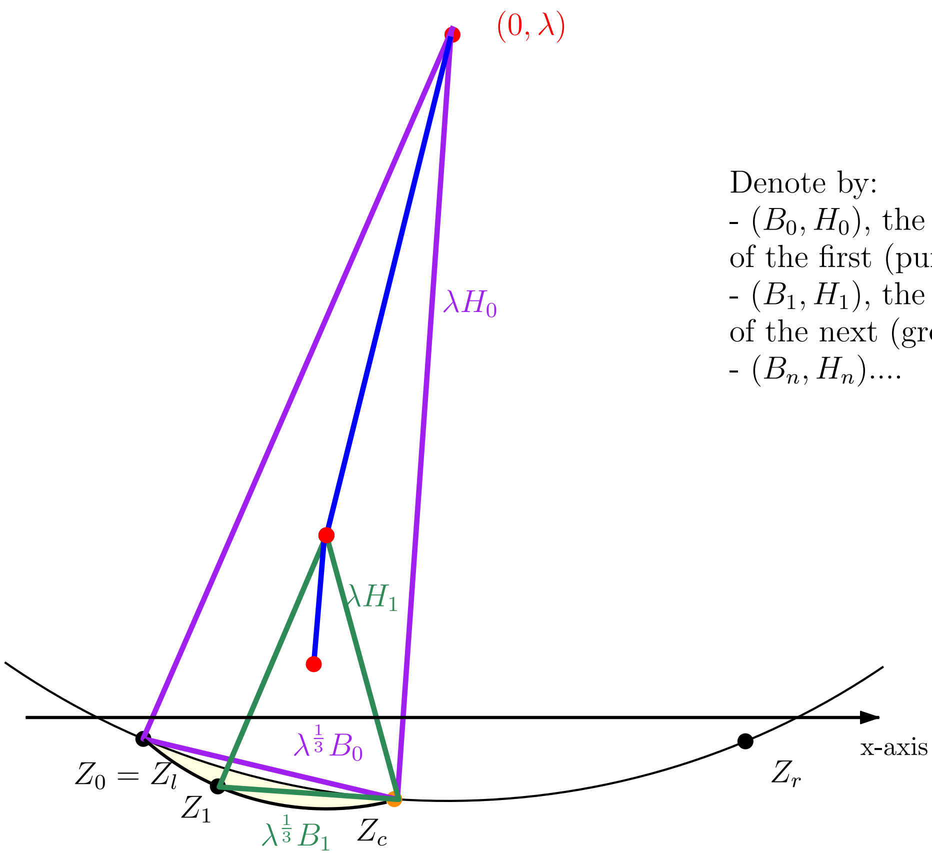

Denote by:
- (B_0, H_0), the basis and height of the first (purple) triangle
- (B_1, H_1), the basis and height of the next (green) triangle
- (B_n, H_n)....

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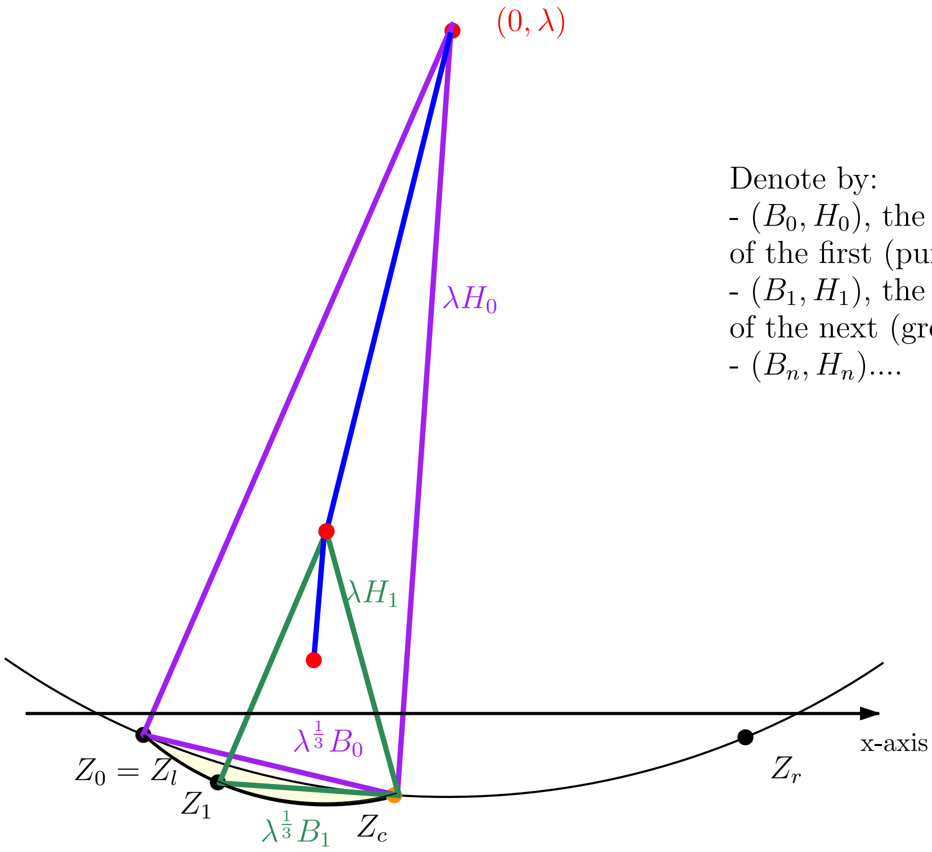

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- (B_0, H_0), the basis and height of the first (purple) triangle
- (B_1, H_1), the basis and height of the next (green) triangle
- (B_n, H_n)....

Proposition. The sequence (B_n, H_n) is a Markov chain which *asymptotically*, for large λ, satisfies the random recursion relation:

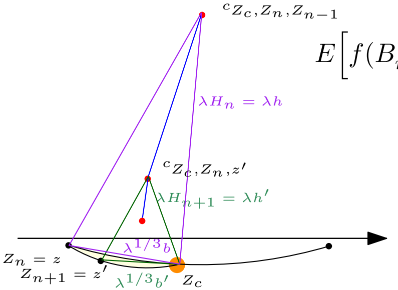
$$(B_{n+1}, H_{n+1}) = \left(\beta_n B_n, \frac{(B_n)^3 H_n}{(B_n)^3 + \frac{3}{2}\xi_n H_n} \right)$$

where $(\beta_n)_n$ and $(\xi_n)_n$ are sequences of iid variables which are respectively $B(2,2)$ and $Exp(1)$ distributed.

Proof: As for the starting point, the proof goes in two steps.

the new couple (b', h') in terms of z'

We apply **Mecke-Slivnyak's formula** for any non-negative measurable function f,



$$E\left[f(B_{n+1}^{(\lambda)}, H_{n+1}^{(\lambda)}) \mid (B_n^{(\lambda)}, H_n^{(\lambda)}) = (b,h)\right] = E\left[\sum_{z' \in \mathcal{P}_{(b,h)}^{(\lambda)}} f(G_{(b,h)}^{(\lambda)}(z'))\mathbf{1}_{\mathcal{V}}(c_{Z_c,Z_n,z'})\right]$$

$$= \int f(G_{(b,h)}^{(\lambda)}(z'))P\left[\mathcal{P}_{(b,h)}^{(\lambda)} \cap \Delta_n^{(\lambda)} = \emptyset\right]dz'$$

$$= \int f(G_{(b,h)}^{(\lambda)}(z'))\mathbf{e}^{-|\boldsymbol{\Delta}_{\mathbf{n}}^{(\lambda)}|}dz'$$

where $\Delta_n^{(\lambda)} = B_{Z_c,Z_n,z'} \setminus B_{Z_c,Z_n,Z_{n-1}}$ is the crescent moon of the picture.

The **change of variable** $z' \mapsto (b', h')$, and the computation of its **Jacobian** finishes the job, and we obtain the following limiting transition density:

$$p^{(\infty)}((b,h),(b',h')) = \frac{\mathbf{4(b-b')b'}}{\mathbf{h'^2}}\exp\left(-\frac{\mathbf{2}}{\mathbf{3}}\mathbf{b^3}\left(\frac{\mathbf{1}}{\mathbf{h'}} - \frac{\mathbf{1}}{\mathbf{h}}\right)\right)\mathbf{1}_{(0,b)}(b')\mathbf{1}_{(0,h)}(h').$$

After some struggle, we deduce the nice probabilistic representation with **Beta and Gamma variables**.

2c. A remarkable property of the Markov chain (B_n, H_n)

Define the **shape quantity** $T_n = \frac{B_n^3}{H_n}$ (denote it is also equal to $\frac{(\lambda^{\frac{1}{3}} B_n)^3}{\lambda H_n}$).

It satisfies the renewal-type recursion relation $T_{n+1} = \beta_n^3 (T_n + \frac{3}{2}\xi_n)$

We can use **Letac's principle** on renewal series to prove the convergence in distribution of the variable T_n. Namely, the variable

$$T_n = \beta_{n-1}^3(\beta_{n-2}^3(\cdots(\beta_0^3(T_0 + \tfrac{3}{2}\xi_0) + \cdots) + \tfrac{3}{2}\xi_{n-1}) \qquad \text{a.s.}$$

has **the same law as** $\beta_0^3(\beta_1^3(\cdots(\beta_{n-1}^3(T_0 + \frac{3}{2}\xi_{n-1}) + \cdots) + \frac{3}{2}\xi_0)$

(putting the indices in reverse order), which is an a.s. convergent sequence.

Therefore the sequence T_n converges in distribution, and we are in one of the very few cases

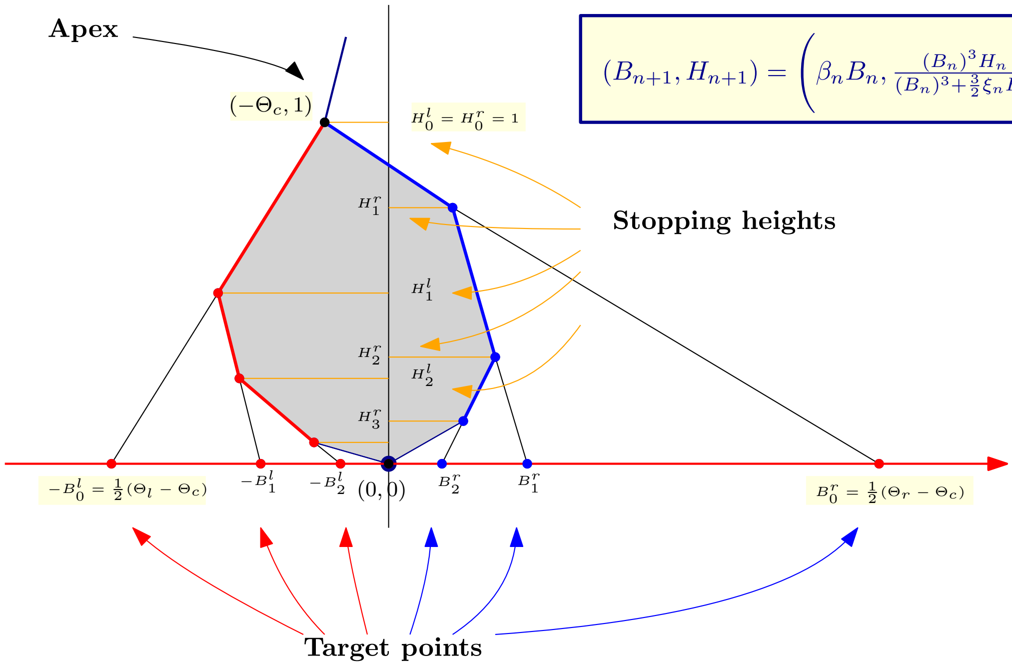
of renewal series having an **explicit limiting law**. Namely,

Proposition. The sequence T_n converges in distribution towards the law of $\frac{3}{2}\Gamma_1^3\Gamma_2\Gamma_3$ where Γ_1, Γ_2 and Γ_3 are independent random variables such that $\Gamma_1 \sim \text{Beta}(2,2)$, $\Gamma_2 \sim \Gamma(\frac{5}{3}, 1)$ and $\Gamma_3 \sim \text{Beta}(2, \frac{2}{3})$ respectively.

2d. The limiting random Menhir

Collecting the two previous results we obtain:

Theorem. After normalization, vertically by λ and horizontally by $\lambda^{\frac{1}{3}}$, a cell of height λ converges in distribution to the **random Menhir** defined below.



$$(B_{n+1}, H_{n+1}) = \left(\beta_n B_n, \frac{(B_n)^3 H_n}{(B_n)^3 + \frac{3}{2}\xi_n H_n}\right)$$

An important part of the work is to **couple** the actual Markov chain $(B_n^{(\lambda)}, H_n^{(\lambda)})$ with the ideal Markov chain along the whole boundary of the cell. For that purpose, one has to introduce large enough regions, which are **stable enough** by the Markov chains, **highly probable** for both Markov chains, and on which both transition kernels are **close in total variation**.

2e. The "curvature" of the Menhir at its base point

Let X_n be the successive first coordinates of the first vertices of the Menhir, so that the n-th vertex has coordinates (X_n, H_n).

Theorem. The sequence $\frac{H_n}{{X_n}^3}$ converges in distribution to a non-degenerate distribution.

Proof. If we introduce, $W_n = \dfrac{B_n^2}{H_n} X_n$, we get the identity $\dfrac{H_n}{X_n^3} = \dfrac{T_n^2}{W_n^3}$.

Now, both T_n and W_n, satisfies contracting recursion relations:

$$W_{n+1} = \beta_n^2 (W_n + \tfrac{3}{2}\xi_n)$$

$$T_{n+1} = \beta_n^3 (T_n + \tfrac{3}{2}\xi_n)$$

As we did for the sequence T_n, we can now use **Letac's principle** on renewal series to prove the convergence in distribution of the couple (T_n, W_n).

Therefore the sequence (T_n, W_n) converges in distribution, and the result follows.

2f. The asymptotic number of vertices of a cell

Theorem. There are asymptotically $\frac{4}{5}\log\lambda$ vertices in a cell of depth λ.

Proof: The initial basis B_0 is of order 1. From the recursion identity,

$$B_n \simeq \beta_0\beta_1\cdots\beta_{n-1}B_0$$

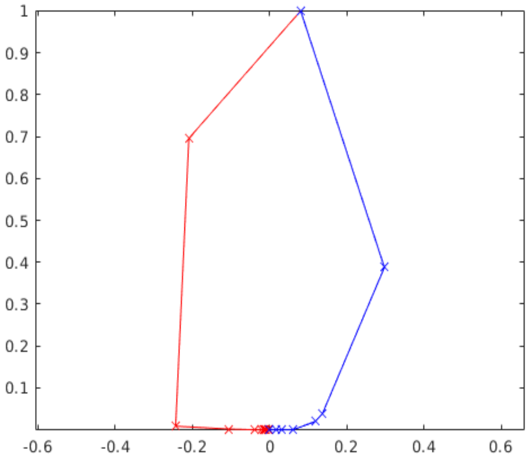
where the variables β_n are iid and $B(2,2)$ distributed.
Since $E[\log\beta_n] = -\frac{5}{6}$,

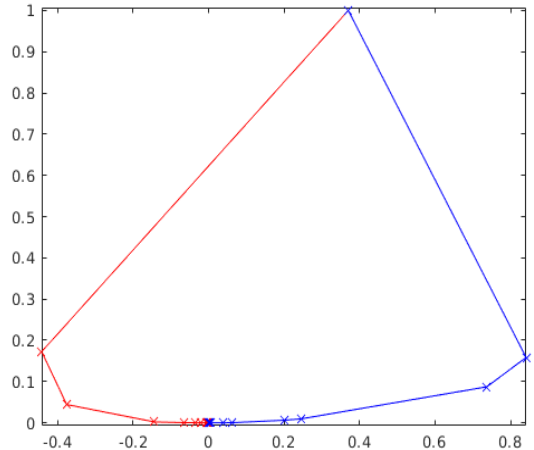
$$B_n = \exp(-\frac{5}{6}n(1+o(1))).$$

At the boundary of the half-plane, the half basis B_n is of order $\lambda^{-\frac{1}{3}}$ since $\lambda^{\frac{1}{3}}B_n$ must be of order 1.
Hence, it takes $\frac{2}{5}\log\lambda$ steps to go all the way down to the boundary of the half-plane.

3. Two simulations

Gallic art remains rather rudimentary:





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