Beyond the mean-field limit for the McKean-Vlasov particle system: Uniform in time estimates for the cumulants

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Today, $\sigma \equiv I_d$ (the identity matrix) to simplify.

A PDE interpretation of the McKean-Vlasov equation

McKean-Vlasov SDE

$$X_t = X_0 + \int_0^t b\big(X_s, \mathcal{L}(X_s)\big)\mathrm{d}s + B_t, \qquad t \geq 0$$

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Starting from μ_0, the flow of marginal laws $(m(t,\mu_0) := \mathcal{L}(X_t))_{t\geq 0}$ satisfies a nonlinear Fokker-Planck equation:

$$\partial_t m(t,\mu) = \frac{1}{2}\triangle m(t,\mu) - \mathrm{div}\Big[m(t,\mu)b(\cdot, m(t,\mu))\Big], \qquad t \geq 0,$$
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This is a *PDE interpretation* of the McKean-Vlasov SDE.

The particle system: first-order, MF interaction and noise

Key question: deriving the previous McKean-Vlasov SDE from a microscopic system. Given N particles (agents), can we obtain the previous SDE as a limit ? In which sense ?

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For particles in the torus $\mathbb{T}^d$:

$$\begin{cases} Y_t^{i,N} = Y_0^{i,N} + \int_0^t b(Y_s^{i,N}, \mu_s^N)\mathrm{d}s + B_t^i, & t \geq 0, \quad 1 \leq i \leq N, \\ \mu_s^N := \frac{1}{N}\sum_{i=1}^N \delta_{Y_s^{i,N}}, \end{cases}$$

where

- $(Y^{i,N})_{i=1}^N$ are the positions;
- μ_s^N empirical measure at time s;
- $b : \mathbb{T}^d \times \mathcal{P}(\mathbb{T}^d) \to \mathbb{R}^d$ is an interaction potential and the mean-field scaling is considered.

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Additionally: smooth b, $(Y_0^{i,N})_{1\leq i\leq N} \sim \mu_0^{\otimes N}$ (i.i.d. initial distributions). The goal is to **relate this particle system to the McKean-Vlasov SDE**.

Assumptions on b

H-stable potential (Carrillo, Gvalani, Pavliotis, Schlichting 2020)

$$b(x,m) = -\kappa \int_{\mathbb{T}^d} \nabla W(x-y)\, m(\mathrm{d}y), \qquad x \in \mathbb{T}^d, m \in \mathcal{P}(\mathbb{T}^d)$$

for $\kappa > 0$ (equal to 1 in what follows for simplicity) and W smooth, coordinate-wise even:

$$W(x_1, \cdots, -x_i, \ldots, x_d) = W(x_1, \ldots, x_i, \ldots, x_d), \quad (x_1, \ldots, x_d) \in \mathbb{T}^d.$$

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Consequence: the Lebesgue measure on $\mathbb{T}^d$, $\mathrm{Leb}_{\mathbb{T}^d}$ is

- ✔ the unique invariant measure for the McKean-Vlasov equation;
- ✔ exponentially stable, i.e. there exists $C, \lambda > 0$ constants s.t. for all $t \geq 0$,

$$\|m(t,\mu) - \mathrm{Leb}_{\mathbb{T}^d}\|_{TV} \leq Ce^{-\lambda t}.$$

Statistical description and mean-field limit

$N \gg 1$, $F_N(t, x_1, \ldots, x_N)$ probability density on the N torus $(\mathbb{T}^d)^N$ at time $t \geq 0$ and F_N^1 first marginal

$$F_N^1(t,z) = \int_{(\mathbb{T}^d)^{N-1}} F_N(t, z, z_2, \ldots, z_N)\mathrm{d}z_2 \ldots \mathrm{d}z_N.$$

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Can we justify this convergence ?

BBGKY, MV version: a formal expansion

F^N solves a forward Kolmogorov equation, where $\bar{x} = (x_1, \ldots, x_N) \in (\mathbb{T}^d)^N$

$$\partial_t F^N(\bar{x}) = \frac{1}{2} \triangle F^N(\bar{x}) + \sum_{i=1}^{N} \mathrm{div}_{x_i} \Big(F^N(\bar{x}) \frac{1}{N} \sum_{j=1}^{N} \nabla W(x_j - x_i) \Big).$$

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Integrating with respect to $x_2, \ldots, x_N$, writing F_N^k for the k-th marginal, and using the coordinate-wise symmetry:

$$\partial_t F_N^1(x) = \frac{1}{2} \triangle F_N^1(x) + \mathrm{div}_x \Big(\int_{\mathbb{T}^d} \nabla W(y - x) F_N^2(x, y) \mathrm{d}y \Big).$$

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Key point: at time 0, $F_N(0, \cdot) = \mu_0^{\otimes N}(\cdot)$ so in particular

$$F_N^2(0, x, y) = F_N^1(0, x) F_N^1(0, y).$$

Not true anymore at time $t > 0$! But one expects $F_N^2(x, y) = F_N^1(x) F_N^1(y) + G_N^2(x, y)$ where G_N^2 is the two-particles correlation function with $G_N^2 \to 0$ as $N \to \infty$.

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Then $\to$ PDE version of the McKean-Vlasov equation on $\mathbb{T}^d$

$$\partial_t f(x) = \frac{1}{2} \triangle f(x) - \mathrm{div}_x \Big(b\big(x, f\big) f(x) \Big).$$

Propagation of chaos I: qualitative convergence

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In general, expect that on $[0,T]$, $T > 0$, and for any fixed $k \in \{1, \dots, N\}$,

$$(Y^{1,N}, \dots, Y^{k,N}) \Longrightarrow (X^1, \dots, X^k),$$

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- For b Lipschitz (w.r.t. the topology of $\mathbb{T}^d \times \mathcal{P}(\mathbb{T}^d)$), Sznitman's coupling, see Sznitman (1991), Lacker (2018)...
- Other approaches: tightness of $(\mathcal{L}(\mu_t^N) \in \mathcal{P}(\mathcal{P}(\mathbb{T}^d)))_{0 \le t \le T}$. Then $(\mathcal{L}(\mu_t^N))_{0 \le t \le T}$ converges weakly to $\delta_{\mathcal{L}(X_t)_{0 \le t \le T}}$.

Propagation of chaos II: strong errors

Two types of results, quantifying strong and weak errors. Second aspect to quantify: uniformity in time ? Strong errors: convergence in some Wasserstein norm, e.g.

$$\sup_{t\geq 0} W_1\Big(F_N^k(t,\cdot), m(t,\mu_0)^{\otimes k}\Big) = O\Big(\frac{1}{N^{\frac{1}{2}}}\Big).$$

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Recent uniform in times results:

- Malrieu 2001 (W convex);
- Durmus-Eberle-Guillin-Zimmer 2020 for small interaction;
- Guillin-Le Bris-Monmarché 2021 for more singular interactions (allowing to treat the Biot-Savart kernel).

Jabin-Wang (2018): non-uniform in time estimates for singular interaction, starting point of several papers.
Other approach to strong error: central limit theorem (Sznitman, Méléard...).

Propagation of chaos III: weak errors

Focus on the statistical behavior of μ_t^N. Goal: deriving rates of convergence (in t and N) for

$$\mathbb{E}\Big[\big|\Phi(\mu_t^N) - \Phi(m(t,\mu_0))\big|\Big],$$

where $\Phi : \mathcal{P}(\mathbb{T}^d) \to \mathbb{R}$ is a test function. Typically Φ is

- ❋ polynomial: Mischler-Mouhot-Wennberg 2015;
- ❋ linear: Bencheikh-Jourdain 2019 (more general b).

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For the torus case, recent results of Delarue-Tse (2021): under regularity assumptions on b and Φ, there exists $C > 0$ such that for all $\mu_0 \in \mathcal{P}(\mathbb{T}^d)$,

$$\sup_{t\geq 0} \mathbb{E}\Big[\big|\Phi(\mu_t^N) - \Phi(m(t,\mu_0))\big|\Big] \leq \frac{C}{N}.$$

Back to the marginals

Recall:

$$\partial_t F_N^1(x) = \frac{1}{2}\triangle F_N^1(x) + \mathrm{div}_x\Big(\int_{\mathbb{T}^d} \nabla W(y-x) F_N^2(x,y)\mathrm{d}y\Big).$$

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Writing $F_N^2(x,y) = F_N^1(x)F_N^1(y) + G_N^2(x,y)$, the previous results show $G_N^2 = O(\frac{1}{N})$ in some weak sense $\to$ McKean-Vlasov equation.

Beyond mean-fields (Bogolyubov corrections?)

What if we keep G_N^2 ? The equation for F_N^1 depending on F_N^2 also writes

$$\partial_t F_N^1(x) = \frac{1}{2}\triangle F_N^1(x) - \mathrm{div}_x\big(b(x, F_N^1)F_N^1(x)\big)$$
$$+ \mathrm{div}_x\Big(\frac{1}{N}\int_{\mathbb{T}^d} \nabla W(x-y)(NG_N^2)(x,y)\mathrm{d}y\Big).$$

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$$\begin{aligned}\partial_t F_N^2(x_1,x_2) = \frac{1}{2}\triangle F_N^2(x_1,x_2) - \sum_{1\le i\ne j\le 2} \mathrm{div}_{x_i}\Big\{ &- \frac{1}{N}\nabla W(x_i - x_j)F_N^1(x_i)F_N^1(x_j) \\ &+ \frac{N-1}{N} b(x_i, F_N^1)F_N^1(x_i)F_N^1(x_j) + 3\frac{N-1}{N} b(x_i, F_N^1)F_N^2(x_i, x_j) \\ &- 3\frac{N-1}{N}\int_{\mathbb{T}^d} \nabla W(x - x_i)F_N^2(x_i, x)\mathrm{d}x F_N^1(x_j) \\ &- 3\frac{N-1}{N}\int_{\mathbb{T}^d} \nabla W(x - x_i)F_N^2(x, x_j)\mathrm{d}x F_N^1(x_1)\Big\} + O\Big(\frac{1}{N^2}\Big).\end{aligned}$$

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Since $G_N^2 = F_N^2 - (F_N^1)^{\otimes 2} \to$ closed form for the evolution of F_N^1 and G_N^2. Initial data

1. $G^2_{N|t=0} = 0$;
2. $F^1_{N|t=0} = \mu_0$.

Controlling the correlations

Expect the contribution of G_N^2 to be of order $O\left(\frac{1}{N}\right)$. With this contribution: correction to this mean-field limit, provided that $G_N^3 = O\left(\frac{1}{N^2}\right)$. And so on...

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Our work: in some weak sense and uniformly in time

$$G_N^{m+1} = O\left(\frac{1}{N^m}\right)$$

for all $m \geq 1$.

A brief reminder on cumulants

(Joint) cumulants of $(Z_1, \dots, Z_n)$ measure the interactions between the variables: for

$$K(t_1, \dots, t_n) = \log \mathbb{E}\Big[e^{\sum_{j=1}^n t_j Z_j}\Big],$$

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$$\kappa^m(X) = \kappa^m(X, \dots, X).$$

A brief reminder on cumulants

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Recall in particular that for all $X \in L^4(\Omega)$,

$$\kappa^2(X) = \mathrm{Var}(X), \qquad \kappa^3(X) = \mathbb{E}\Big[(X - \mathbb{E}[X])^3\Big].$$

But of course it is not always that easy

$$\kappa^4(X) = \mathbb{E}\Big[(X - \mathbb{E}[X])^4\Big] - 3\mathrm{Var}(X)^2.$$

Main result

Our main result is the following:

Theorem (B.-Duerinckx 2022$^+$)

Assume that b is given by a smooth, H-stable potential W, and that $\Phi : \mathcal{P}(\mathbb{T}^d) \to \mathbb{R}$ is smooth. Then, for all $m \geq 1$, there exists a constant $C > 0$ such that, for any $\mu_0 \in \mathcal{P}(\mathbb{T}^d)$,

$$\sup_{t \geq 0} \kappa^{m+1}\left[\Phi(\mu_t^N)\right] \leq \frac{C}{N^m}.$$

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- ✍ Explicit dependency of C in the derivatives of Φ.

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Possible to relate $\kappa^{m+1}[\Phi(\mu_t^N)]$ to the norm of G_N^{m+1} when $\Phi(\mu) = \int_{\mathbb{T}^d} \varphi(x)\mu(\mathrm{d}x)$ with φ smooth.

The sources of randomness

Recall $\mu_t^N = \frac{1}{N}\sum_{i=1}^N \delta_{Y_t^{i,N}}$ for all $t \geq 0$. Let $\Phi : \mathcal{P}(\mathbb{T}^d) \to \mathbb{R}$. Weak formulation of the result for G_N^2:

$$\mathrm{Var}\Big[\Phi(\mu_t^N)\Big] = O\Big(\frac{1}{N}\Big),$$

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- Brownian motions;
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- ☆ $\mathbb{E}_B[\Phi(\mu_t^N)] = \Phi(m(t, \mu_0^N)) + O\Big(\frac{1}{N}\Big)$;
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In both cases, ergodic estimates to obtain the uniform control in time.

Linear functional derivatives

Let $F : \mathcal{P}(\mathbb{T}^d) \to \mathbb{R}$. We say that F is continuously differentiable if there exists a continuous function $\frac{\delta F}{\delta m} : \mathcal{P}(\mathbb{T}^d) \times \mathbb{T}^d \to \mathbb{R}$ such that, for any $\mu, \mu' \in \mathcal{P}(\mathbb{T}^d)$,

$$F(\mu) - F(\mu') = \int_0^1 \int_{\mathbb{T}^d} \frac{\delta F}{\delta m}\big(s\mu + (1-s)\mu', y\big)(\mu - \mu')(\mathrm{d}y)\mathrm{d}s.$$

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Let $\gamma : (\mathbb{T}^d)^N \to \mathbb{R}$. Glauber derivative with respect to $Y_0^{1,N}$:

$$D_\circ^1[\gamma(Y_0^{1,N}, \ldots, Y_0^{N,N})] = \gamma(Y_0^{1,N}, \ldots, Y_0^{N,N}) - \int_{\mathbb{T}^d} \gamma(z, \ldots, Y_0^{N,N}) \mu_0(\mathrm{d}z)$$

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For any $\psi : \mathcal{P}(\mathbb{T}^d) \to \mathbb{R}$ admitting linear derivative, any $j \in [N]$,

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Efron-Stein's inequality:

$$\mathrm{Var}_\circ[Y] \leq \mathbb{E}^\circ\Big[\sum_{j=1}^N \big|D_\circ^j[Y]\big|^2\Big].$$

Similar *Poincaré inequality* for higher-order cumulants.

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The master equation

For any $\Phi : \mathcal{P}(\mathbb{T}^d) \to \mathbb{R}$, write $\mathcal{U}_\Phi(t,\mu) = \Phi(m(t,\mu))$ for $t \geq 0$, $\mu \in \mathcal{P}(\mathbb{T}^d)$. Then, from Buckdahn-Li-Peng-Rainer (2017), $\mathcal{U}_\Phi$ satisfies the master equation

$$\begin{cases} \partial_t \mathcal{U}_\Phi(t,\mu) & = \int_{\mathbb{T}^d} \Big[\sum_{i=1}^d \partial_{x_i} \frac{\delta \mathcal{U}_\Phi}{\delta m}(t,\mu,x) b_i(x,\mu) \\ & \qquad + \frac{1}{2} \sum_{i,j=1}^d \partial^2_{x_i x_j} \frac{\delta \mathcal{U}_\Phi}{\delta m}(t,\mu,x) \Big] \mu(\mathrm{d}x) \quad t \geq 0, \\ \mathcal{U}_\Phi(0,\mu) & = \Phi(\mu) \end{cases}$$

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$\to$ expand $m(t,\mu)$ along the dynamics. From Chassagneux-Szpruch-Tse (2019), we have

$$\mathbb{E}_B[\Phi(\mu^N_t)] = \mathcal{U}_\Phi(t,\mu^N_0) + \frac{1}{2N} \int_0^t \int_{\mathbb{T}^d} \mathbb{E}_B \Big[\mathrm{Tr}[\partial^2_\mu \mathcal{U}_\Phi(t-s,\mu^N_s,v,v) \Big] \mu^N_s(\mathrm{d}v)\mathrm{d}s,$$

where $\partial_\mu \mathcal{U}_\Phi(t-s,\mu,y) = \partial_y \frac{\delta \mathcal{U}}{\delta m}(t-s,\mu,y)$.

Pushing the expansion further

Set, for $0 \leq s \leq t$, $\mu \in \mathcal{P}(\mathbb{T}^d)$,

$$\Phi^{(1)}\big((t,s),\mu\big) = \int_{\mathbb{T}^d} \mathrm{Tr}\Big[\partial_\mu^2 \mathcal{U}_\Phi\big(t-s,\mu,y,y\big)\Big]\mu(\mathrm{d}y),$$

and then set, for $0 \leq u \leq s \leq t$,

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Explicit formulas relating $\partial_\mu^2 \mathcal{U}_\Phi^{(1)}$ with Wasserstein derivatives of Φ evaluated at solutions of linearized parabolic equations. In particular, using ergodic estimates for those solutions:

$$\sup_{\mu\in\mathcal{P}(\mathbb{T}^d)} \int_{\mathbb{T}^d}\int_0^t\int_0^s \mathrm{Tr}\Big[\partial_\mu^2 \mathcal{U}_\Phi^{(1)}\big((t,s,u),\mu,y,y\big)\Big]\mu(\mathrm{d}y)\mathrm{d}u\mathrm{d}s = O(1).$$

Treating the Brownian cumulants

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\begin{aligned}
\mathrm{Var}_B[\Phi(\mu_t^N)] &= \mathbb{E}_B[\Phi(\mu_t^N)^2] - \mathbb{E}_B[\Phi(\mu_t^N)]^2 \\
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and $\mathcal{U}_{\Phi^2}(t,\mu_0^N) = \Phi^2(m(t,\mu_0^N)) = \mathcal{U}_{\Phi}(m(t,\mu_0^N))^2$ so $\mathrm{Var}_B[\Phi(\mu_t^N)] = O\Big(\frac{1}{N}\Big)$.

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We can do much more ! Identifying precisely the $O\Big(\frac{1}{N}\Big)$ term: since

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\partial_\mu^2 f^2(\nu)(x,x) = 2f(\nu)\big(\partial_\mu^2 f(\nu)(x,x)\big) + 2\big(\partial_\mu f(\nu)(x)\big)^2
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Hence,

$$\mathrm{Var}_B[\Phi(\mu_t^N)] = \frac{1}{N}\int_0^t \int_{\mathbb{T}^d} \Big|\partial_\mu \mathcal{U}_\Phi(t-s, m(s,\mu_0^N), y)\Big|^2 m(s,\mu_0^N)(\mathrm{d}y)\mathrm{d}s + O\Big(\frac{1}{N^2}\Big).$$

Can apply Glauber calculus to this leading term !

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Thank you for your attention !