Asymptotic behavior of cumulative processes and application to Hawkes processes

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1. Cumulative processes
   - Definition and examples
   - Long-time behavior

2. Application to Hawkes Processes
   - Definition of Hawkes Processes and construction
   - Link with cumulative processes and results

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2. Application to Hawkes Processes

Renewal process

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Let $(\tau_i)_i$ i.i.d random variables in $\mathbb{R}^+$.
Let S_n the cumuled sum: $S_n = \sum_{i=1}^n \tau_i$ and M_t the counting process:

$$M_t = \sup_{n \in \mathbb{N}} \{S_n \leq t\}.$$

M_t is called the *renewal process* of $(\tau_i)_i$.

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Example

If $\tau_i \sim \mathcal{E}(\lambda)$, M_t is a Poisson process of parameter λ.

Simple definition of cumulative process

Definition

Let $(\tau_i, W_i)_i$ i.i.d. couples of random variables in $\mathbb{R}^+ \times \mathbb{R}$.
Let M_t the counting process associated with $(\tau_i)_i$:

$$M_t = \sup_{n \in \mathbb{N}} \left\{ \sum_{i=1}^{n} \tau_i \leq t \right\}.$$

The *cumulative process* associated with $(\tau_i, W_i)_i$ is

$$Z_t = \sum_{i=1}^{M_t} W_i.$$

Complete definition

Definition

Z_t a real valued processes such that:

- $Z_0 = 0$
- there exists a renewal process S_n, s.t. for any n, $(Z_{S_n+t} - Z_{S_n})$ is independent of $S_0, \dots, S_n$ and $(Z_s)_{s<S_n}$
- the distribution of $(Z_{S_n+t} - Z_{S_n})$ is independent of n.

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Remark

We can write

$$Z_t = W_0(t) + W_1 + \dots + W_{M_t} + r_t,$$

where M_t is defined by $M_t = \sup\{n \geq 0, S_n \leq t\}$; and $\tau_n = S_n - S_{n-1}$ for $n \geq 1$. Thus, $(\tau_k, W_k)_k$ are i.i.d.

Examples

Example 1

Poisson process, and renewal process in general.

[1] Raphaël Lefevere, Mauro Mariani, and Lorenzo Zambotti. "Large Deviations for Renewal Processes". In: *Stochastic Processes and their Applications* 121.10 (2011), pp. 2243–2271.

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Example 2

If $W_i = F(\tau_i)$, where F is a deterministic function[1].

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Example 3

Hawkes processes in a specific context (see later).

[1] Raphaël Lefevere, Mauro Mariani, and Lorenzo Zambotti. "Large Deviations for Renewal Processes". In: *Stochastic Processes and their Applications* 121.10 (2011), pp. 2243–2271.

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2. Application to Hawkes Processes

Law of large numbers, Central Limit Theorem[2]

Law of Large Numbers

If $\mathbb{E}[|W|]$ and $\mathbb{E}[\tau]$ are finite, and under conditions on the law of r_t, we have

$$\frac{Z_t}{t} \xrightarrow[t\to\infty]{a.s.} \frac{\mathbb{E}[W]}{\mathbb{E}[\tau]}.$$

[2] Soeren Asmussen. *Applied Probability and Queues*. 2nd ed. Stochastic Modelling and Applied Probability. New York: Springer-Verlag, 2003.

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Central Limit Theorem

If $\mathrm{Var}(W) < \infty$ and $\mathrm{Var}(\tau) < \infty$, then

$$\frac{\left(Z_t - t\frac{\mathbb{E}[W]}{\mathbb{E}[\tau]}\right)}{\sqrt{t}} \xrightarrow[t\to\infty]{} \mathcal{N}\left(0,\sigma^2\right) \text{ where } \sigma^2 = \frac{1}{\mathbb{E}(\tau)}\mathrm{Var}\left(W - \frac{\mathbb{E}[W]}{\mathbb{E}[\tau]}\tau\right)$$

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Large deviations inequalities

We denote $\theta_0 = \sup\{\theta, \mathbb{E}(e^{\theta W}) < \infty\}$ and $m = \mathbb{E}(W_1)/\mathbb{E}(\tau_1)$.

Large deviations inequalities [Cattiaux, C., Costa]

For all $a > 0$

$$\limsup_{t\to+\infty} \frac{1}{t} \ln \mathbb{P}\left(\frac{1}{t}\sum_{i=1}^{M_t} W_i \geq m + a\right) \leq -\min\left[\inf_{z\geq m+(a/2)} J(z)\ ,\ \theta_0 a/4\right],$$

and

$$\limsup_{t\to+\infty} \frac{1}{t} \ln \mathbb{P}\left(\frac{1}{t}\sum_{i=1}^{M_t} W_i < m - a\right) \leq -\min\left[\inf_{z\leq m-(a/2)} J(z)\ ,\ \theta_0 a/4\right].$$

Large deviations inequalities

Here Λ^* is the Cramer transform for $(a,b) \in \mathbb{R}^2$, and J the rate function associated for $z \in \mathbb{R}^+$

$$\Lambda^*(a,b) = \sup_{x,y} \left\{ ax + by - \ln\left(\mathbb{E}\left[e^{x\tau + yW}\right]\right)\right\} \quad \text{and} \quad J(z) = \inf_{\beta>0} \beta\,\Lambda^*\left(\frac{1}{\beta}, \frac{z}{\beta}\right).$$

Large deviations principle

Large deviation principle [Cattiaux, C., Costa]

If $\theta_0 = +\infty$ (in particular, if W is bounded) then $\frac{1}{t}\sum_{i=1}^{M_t} W_i$ satisfies a full LDP with a good rate function $\tilde{J}$, i.e.

$$\text{for any closed set } \mathcal{C} \in \mathbb{R}, \qquad \limsup_{t\to\infty} \frac{1}{t} \ln \mathbb{P}\left(\frac{1}{t}\sum_{i=1}^{M_t} W_i \in \mathcal{C}\right) \leq -\inf_{m\in\mathcal{C}} \tilde{J}(m),$$

$$\text{for any open set } \mathcal{O} \in \mathbb{R}, \qquad \liminf_{t\to\infty} \frac{1}{t} \ln \mathbb{P}\left(\frac{1}{t}\sum_{i=1}^{M_t} W_i \in \mathcal{O}\right) \leq -\inf_{m\in\mathcal{O}} \tilde{J}(m).$$

Large deviations principle

Rate functions

For W^n a well-chosen reduction of W, we introduce the Cramer transform for $(a, b) \in \mathbb{R}^2$, and the rate function J^n associated for $z \in \mathbb{R}^+$

$$\Lambda_n^*(a, b) = \sup_{x,y} \left\{ ax + by - \ln \mathbb{E}\left(e^{x\tau + yW^n}\right)\right\} \quad \text{and} \quad J^n(z) = \inf_{\beta > 0} \beta \Lambda_n^* \left(\frac{1}{\beta}, \frac{z}{\beta}\right).$$

We define

$$\tilde{J}(z) = \sup_{\delta > 0} \liminf_{n \to \infty} \inf_{|y - z| < \delta} J^n(y).$$

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Definition

Definition

Let $h : (0, +\infty) \to \mathbb{R}$ a signed measurable function. Let $\lambda > 0$.
A Hawkes process N^h is a self-influencing point process whose intensity is given at each time $t \geq 0$ by:

$$\Lambda^h(t) = \left(\lambda + \sum_{i \geq 1} h(t - U_i)\right)^+ = \left(\lambda + \int_{(-\infty,t)} h(t-u) N^h(du)\right)^+$$

where U_i are the jumps of N^h.

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Applications

- Earthquakes and aftershocks
- Firing of neurons
- Social network
- Finance

Construction (example)

Example

$\lambda = 1$

$h = \mathbb{1}_{[0,0.5]} - \mathbb{1}_{(0.5,1]}$

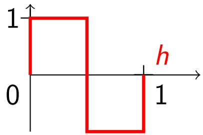

Figure: function h

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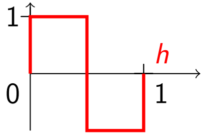

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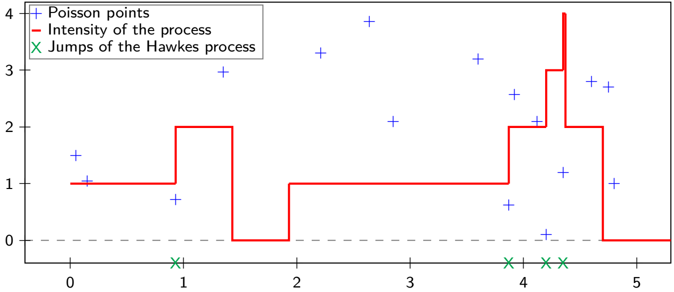

Figure: N^h and its intensity in function of the time

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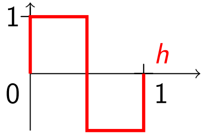

Figure: function h

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Construction from an SDE

Proposition

Let Q a $(\mathcal{F}_t)_{t\geq 0}$-Poisson point process on $(0,+\infty)^2$ with unit intensity. Let $\lambda > 0$ and $h : (0,\infty) \to \mathbb{R}$ a signed reproduction function with $\|h^+\|_1 < 1$. Then, there exists a unique strong solution of:

$$\begin{cases} N^h = \int \delta_u \mathbb{1}_{\theta \leq \Lambda^h(u)} Q(du, d\theta) \\ \Lambda^h(u) = \left(\lambda + \int_{(-\infty,u)} h(u-s) N^h(du)\right)^+, \ u > 0, \end{cases}$$

and this solution is a Hawkes process.

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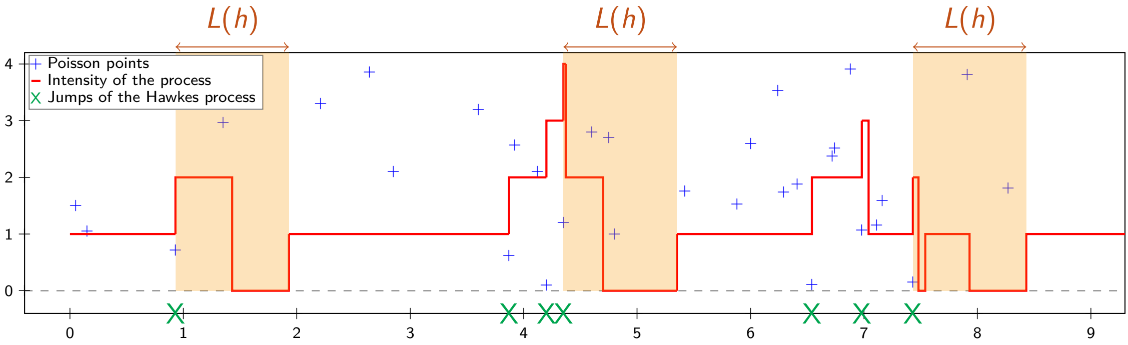
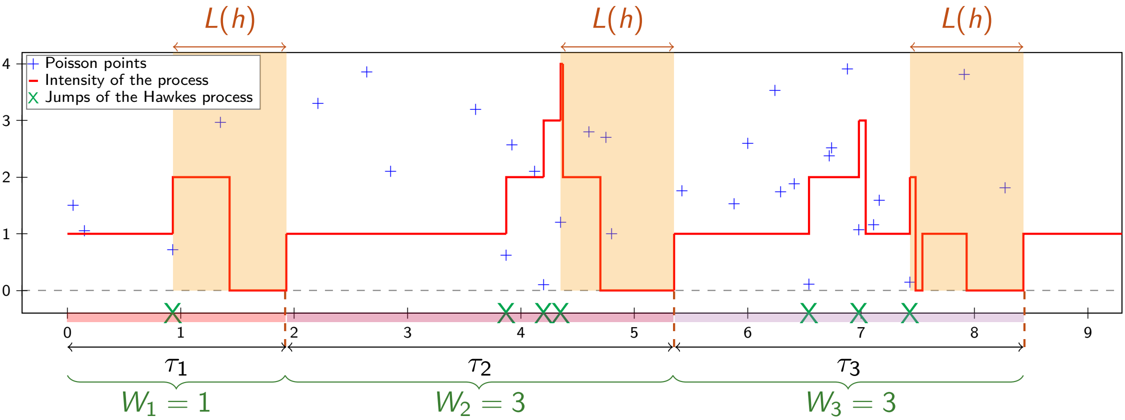
Link between Hawkes process and cumulative process

Assumption: h **has a compact support** included in $[0, L(h)]$.
Then, the Hawkes process is a cumulative process.

Intensity

$\Lambda^h(t) = \left(\lambda + \sum_{i \geq 1} h(t - U_i)\right)^+$, where U_i are the jumps of N^h.
If $t > U_i + L(h)$ for each $U_i < t$, then $\Lambda^h(t) = \lambda$.


L(h)
L(h)
L(h)
Poisson points
Intensity of the process
Jumps of the Hawkes process
0
1
2
3
4
0
1
2
3
4
5
6
7
8
9



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1
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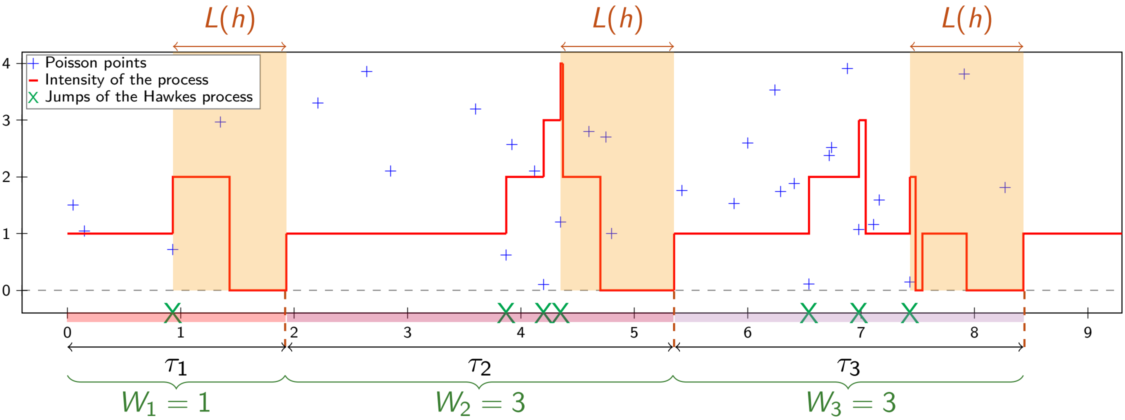
τ_1
τ_2
τ_3
$W_1 = 1$
$W_2 = 3$
$W_3 = 3$



$\tau_1 = \inf\{t > U_1, N^h((t - L(h), t]) = 0\}$

$W_1 = N^h([0, \tau_1])$.

$N^h(t) = \sum_{i=1}^{M_t} W_i + R_t$

where $0 \leq R_t \leq W_{M_t+1}$

Results : law of large numbers and limit central theorem

Law of large numbers and Limit Central Theorem for Hawkes process [Cattiaux, C., Costa]

Let h be a signed function, with a support includes in $[0, L(h)]$. Then we have:

$$\frac{N_t^h}{t} \xrightarrow[t\to\infty]{a.s.} \frac{\mathbb{E}[W_1]}{\mathbb{E}[\tau_1]}$$

and

$$\frac{1}{\sqrt{t}}\left(N_t^h - \frac{\mathbb{E}[W_1]}{\mathbb{E}[\tau_1]}t\right) \xrightarrow[t\to\infty]{law} \mathcal{N}(0,\sigma^2),$$

where $\sigma^2 = \frac{1}{\mathbb{E}(\tau_1)} Var\left(W_1 - \frac{\mathbb{E}[W_1]}{\mathbb{E}[\tau_1]}\tau_1\right)$.

Results: large deviations

Let θ_0 such that $\forall \theta < \theta_0, \mathbb{E}(e^{\theta|W|}) < \infty$.

Large deviations inequalities [Cattiaux, C., Costa]

- If $\theta_0 = \infty$, then N_t^h/t satisfies a LDP.

Results: large deviations

Let θ_0 such that $\forall \theta < \theta_0, \mathbb{E}(e^{\theta|W|}) < \infty$.

Large deviations inequalities [Cattiaux, C., Costa]

- If $\theta_0 < +\infty$, we have for all $a > 0$

$$\limsup_{t\to\infty} \frac{1}{t} \ln \mathbb{P}\left(\frac{N_t^h}{t} > m + a\right) \leq - \min\left[\inf_{z \geq m+a/2} J(z)\, ,\, \theta_0 a/4\right],$$

$$\limsup_{t\to\infty} \frac{1}{t} \ln \mathbb{P}\left(\frac{N_t^h}{t} < m - a\right) \leq - \min\left[\inf_{z \leq m-a/2} J(z)\, ,\, \theta_0 a/4\right],$$

where we define

$$\Lambda^*(a,b) = \sup_{x,y}\left\{ax + by - \ln\left(\mathbb{E}\left[e^{x\tau + yW}\right]\right)\right\} \quad \text{and} \quad J(z) = \inf_{\beta > 0} \beta\, \Lambda^*\left(\frac{1}{\beta}, \frac{z}{\beta}\right).$$

Thank you

- Patrick Cattiaux, Laetitia Colombani, and Manon Costa. *Large Deviation Principles for Cumulative Processes and Applications*. 2021. arXiv: 2109.07800
- Patrick Cattiaux, Laetitia Colombani, and Manon Costa. "Limit Theorems for Hawkes Processes Including Inhibition". In: *Stochastic Processes and their Applications* (2022)

Idea of the proof

- Simplify the problem : $W \in \{1, \ldots, N\}$

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- Focus on the empirical measure

$$\mu_t = \frac{1}{t} \int_0^t \delta_{(s - S_{M_s}, S_{M_s+1} - s, W_{M_s+1})} ds$$

1. Build an interesting space of measures and determine a first rate function.
2. Prove the upper bound for compact spaces, by constructing the rate function; and extend it to closed spaces.
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- Use a contraction principle to deduce a LDP on Z_t/t in the simplified case.
- Study the behavior of the leftover between the approximation and the "true" process.
- Complete the work with a study of rate function