Dynamical Random Graphs: Local Convergence Point of View

Based on a joint work with Emmanuel Jacob (UMPA, ÉNS de Lyon)

Léo Dort

Unité de Mathématiques Pures et Appliquées (ÉNS de Lyon)

Journées MAS 2022

August 30th 2022, Rouen

Let's Begin

INTRODUCTION

- Motivations
- Contents

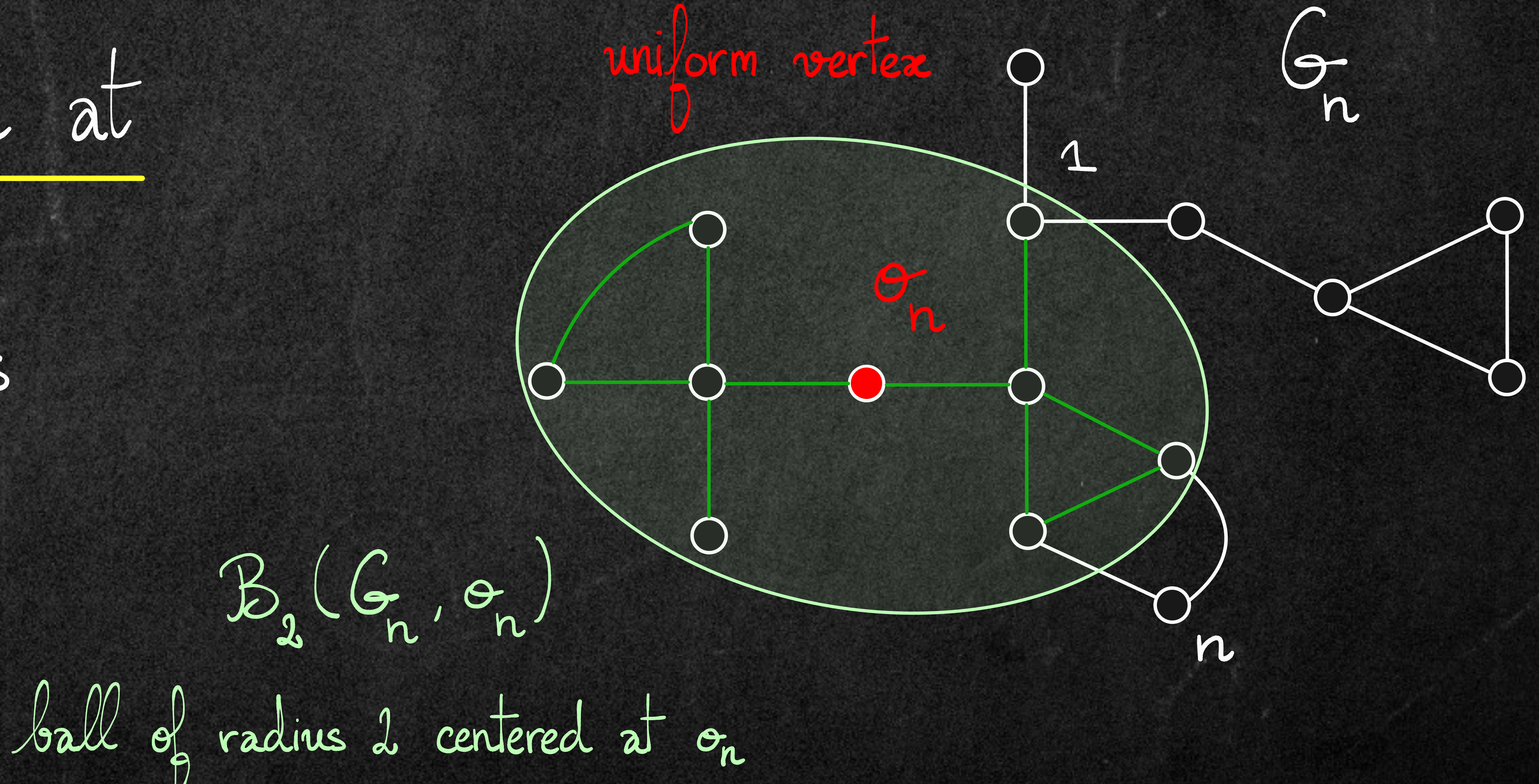
Local Weak Convergence of Graphs

We look at (dynamical) graphs from a local point of view :

⇝ consider the distribution of the neighbourhood of a random vertex

Itaï Benjamini & Oded Schramm, 2001

Consider a graph $G_n = (V_n, E_n)$ with n vertices $V_n = \{1, \dots, n\}$, and study the geometry of the ball centered at uniform vertex when the number of vertices tends to ∞.



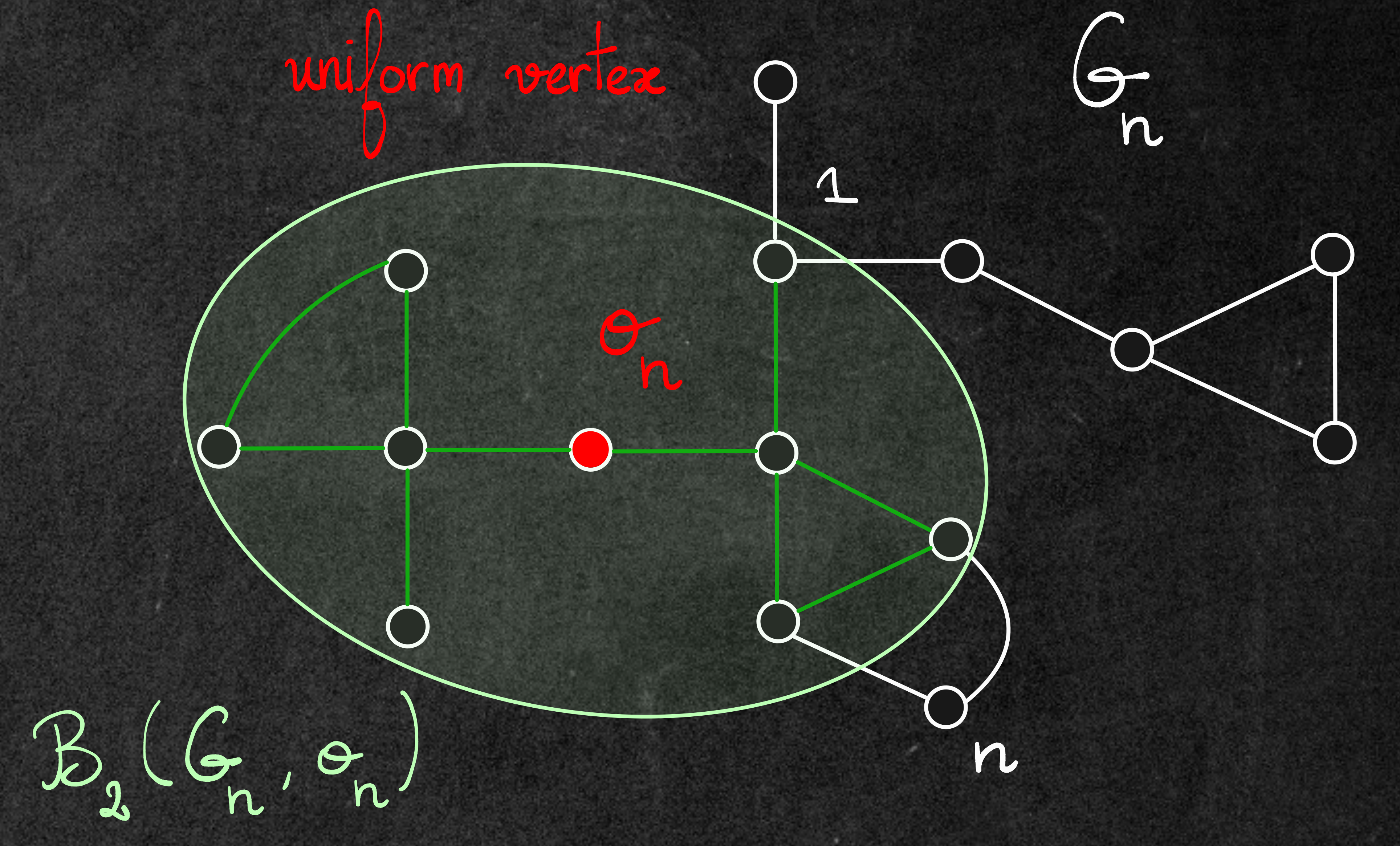
Local Weak Convergence of Graphs

We say that

$$G_n \xrightarrow[n \to +\infty]{\text{loc.}} (G, o)$$

with n vertices

infinite random locally finite rooted graphs

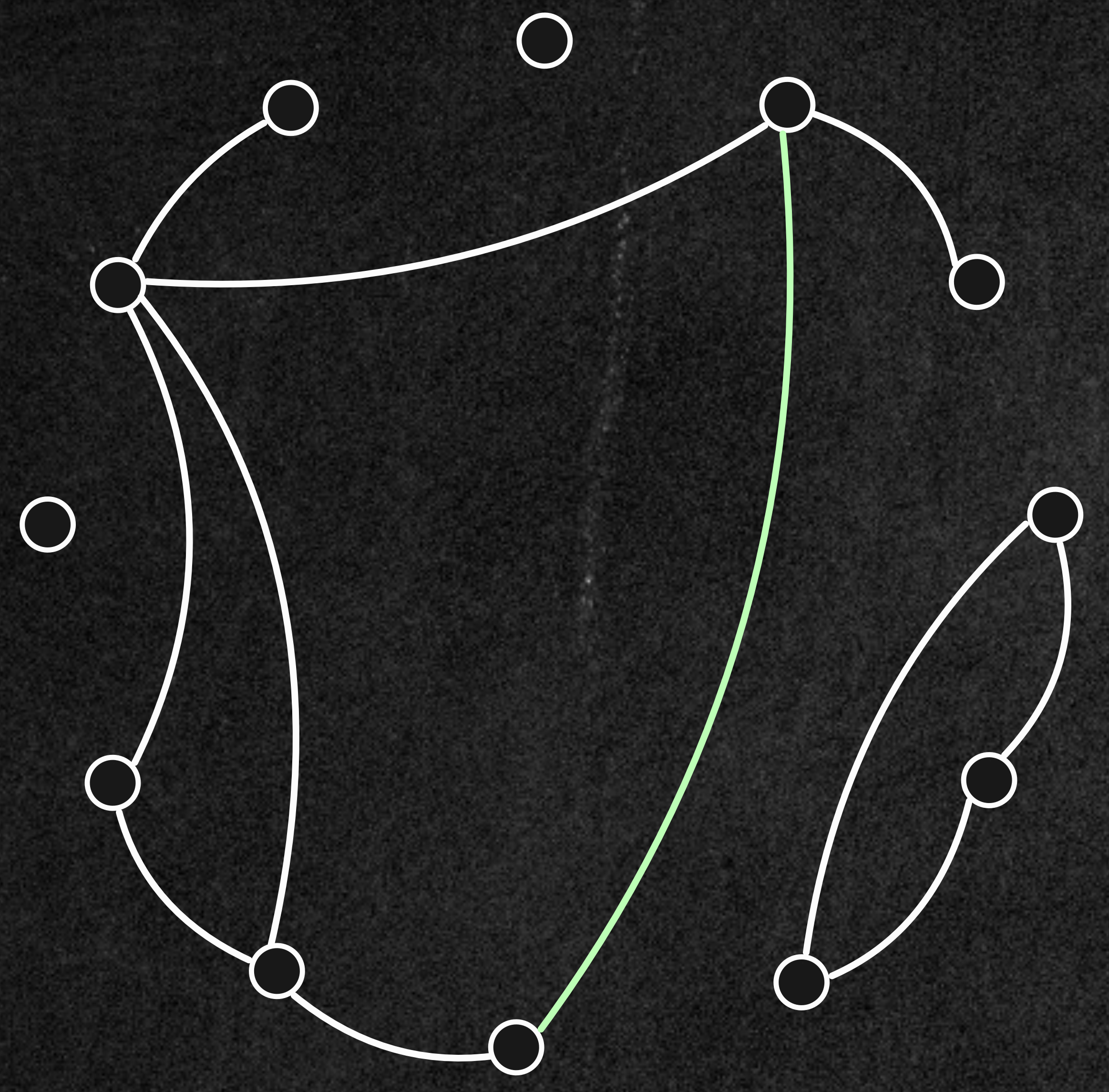


if for every $d \geq 1$:

$$\frac{1}{|V_n|} \sum_{v \in V_n} \mathbb{1}_{\{B_d(G_n, v) \equiv \bullet\}} \xrightarrow[n \to +\infty]{} \mathbb{P}\left(B_d(G, o) \equiv \bullet \right)$$

⇝ (G, o) describes the local geometry of G_n around a random node.

An Important Example: The Erdös-Rényi Random Graph



ER with n nodes

and probabilty of edges $\frac{\kappa}{n}$

Gilbert (1959)

Erdös & Rényi (1959)

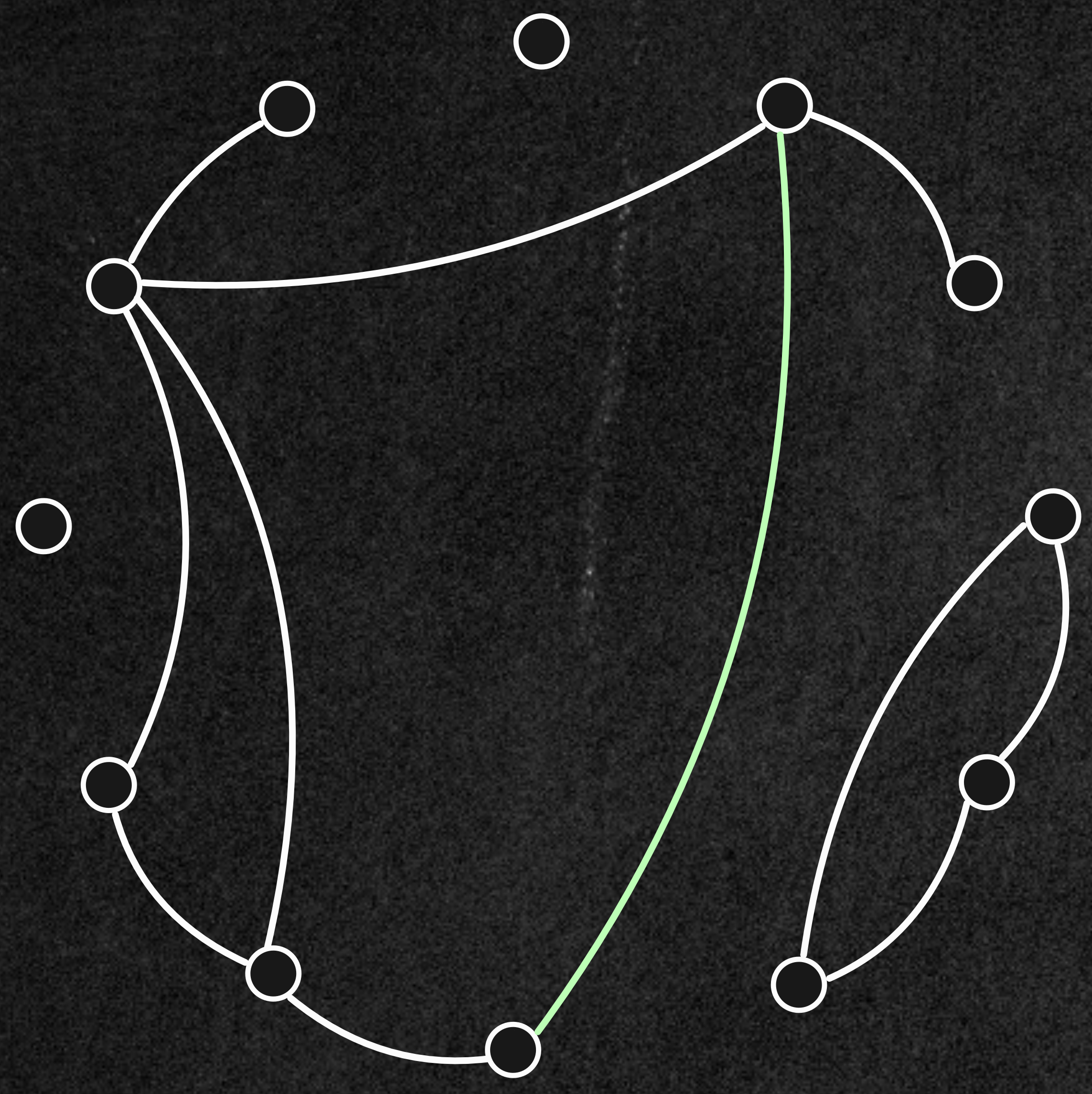
- **Locally tree-like structure**: for each $\ell \geq 3$, the number of cycle with length ℓ in ER is a $o_{\mathbb{P}}(n)$.

- **Binomial converges to Poisson**: degree of each vertex is a binomial r.v. $\text{Bin}\left(n-1, \frac{\kappa}{n}\right)$:

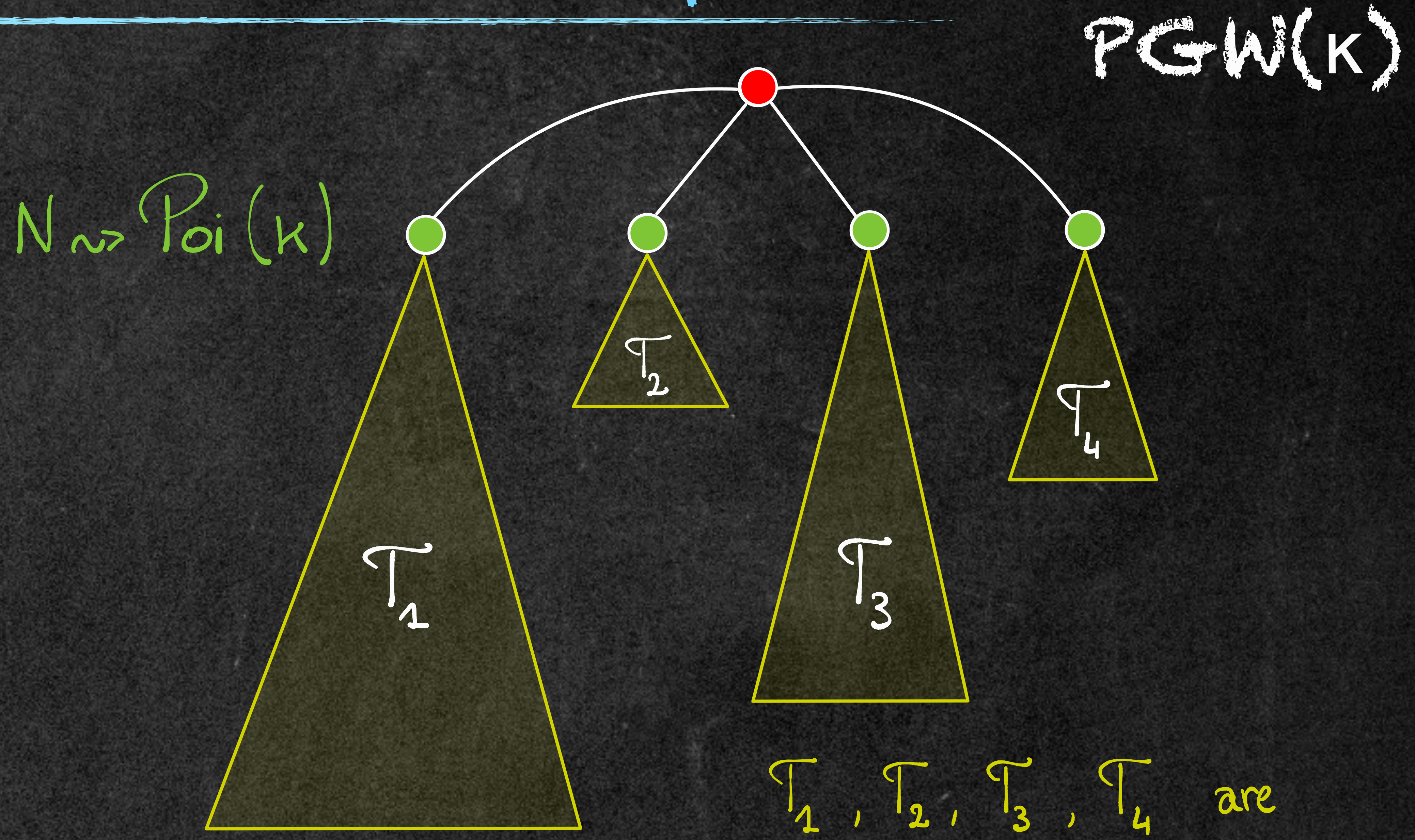
$$\mathbb{P}\left[\deg_{\mathcal{G}(n,\frac{\kappa}{n})}(o) = k\right] = \binom{n-1}{k} \cdot \left(\frac{\kappa}{n}\right)^{k} \cdot \left(1 - \frac{\kappa}{n}\right)^{n-1-k}$$

$$\xrightarrow[n \to +\infty]{} e^{-\kappa} \frac{\kappa^{k}}{k!} = \mathbb{P}\left[\text{Poi}(\kappa) = k\right].$$

An Important Example: The Erdös-Rényi Random Graph



$\xrightarrow[n \longrightarrow +\infty]{\text{loc}}$



T_1, T_2, T_3, T_4 are independent with law PGW(κ)

ER with n nodes and probabilty of edges $\frac{\kappa}{n}$

Gilbert (1959)

Erdös & Rényi (1959)

- Locally tree-like structure : for each $\ell \geq 3$, the number of cycle with length ℓ in ER is a $o_{\mathbb{P}}(n)$.

- Binomial converges to Poisson : degree of each vertex is a binomial r.v. $\mathrm{Bin}\left(n-1, \frac{\kappa}{n}\right)$:

$$\mathbb{P}\left[\deg_{\mathcal{G}(n,\frac{\kappa}{n})}(o) = k\right] = \binom{n-1}{k} \cdot \left(\frac{\kappa}{n}\right)^k \cdot \left(1 - \frac{\kappa}{n}\right)^{n-1-k}$$

$$\xrightarrow[n \to +\infty]{} e^{-\kappa} \frac{\kappa^k}{k!} = \mathbb{P}\left[\mathrm{Poi}(\kappa) = k\right].$$

Contents

LOCAL WEAK LIMIT OF DYNAMICAL GRAPHS

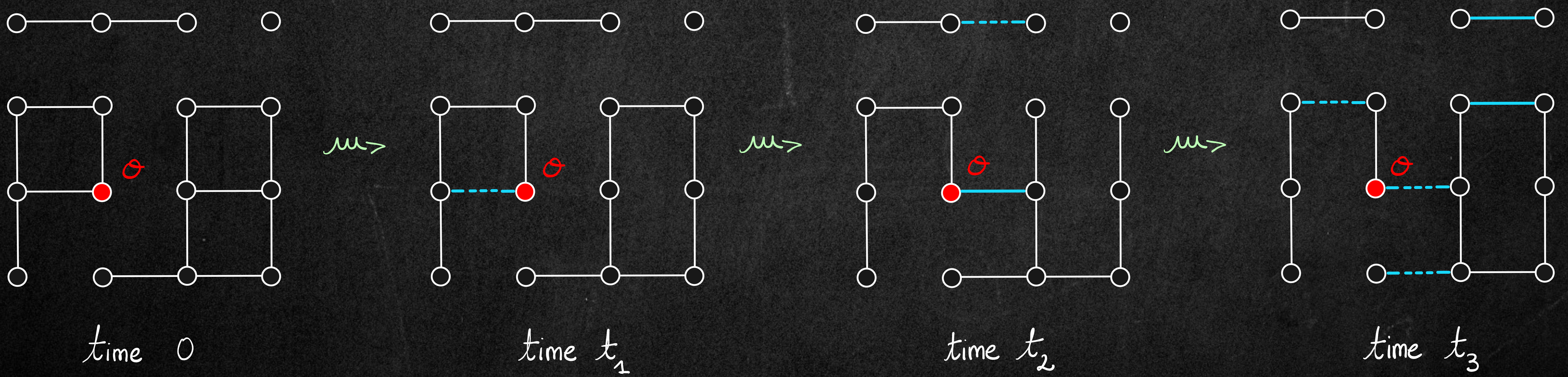
- Dynamical Balls
- Local Convergence

Some Definitions

Def: A **DYNAMICAL ROOTED GRAPH** on a vertex set $\mathcal{V}$ (stack of vertices) is a process of rooted graphs $(G_t, o)_{t\geq 0}$ such that :

$G_t = (V_t, E_t)$ — sets of vertices & edges at time t

$o \in V_0$ and $\forall\, t, s \geq 0,\ V_t \subseteq V_{t+s} \subseteq \mathcal{V}$.



Some Definitions

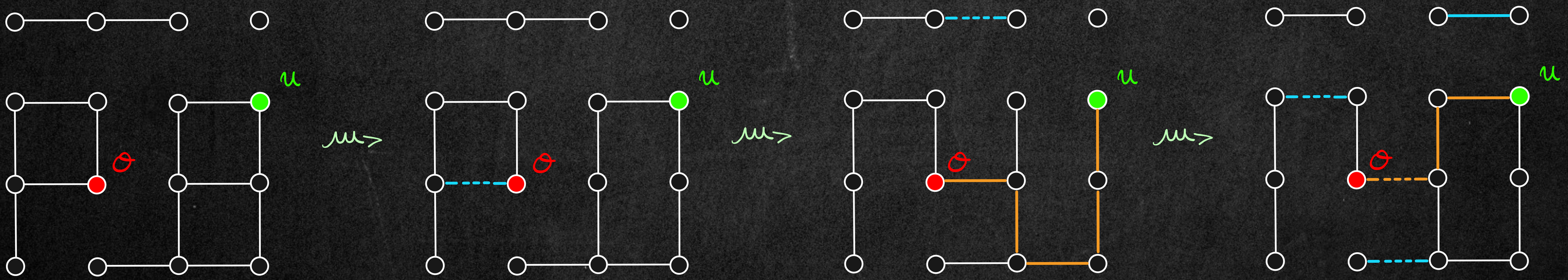
Def: If $0 \le s \le t$, $u \in V_s$ and $v \in V_t$, we call **WEAK DISTANCE** from u to v in $[s,t]$:

$$\vec{d}_{s,t}(u,v) := \inf \Big\{ k \ge 0 \;:\; \exists\, s \le r_1 \le \cdots \le r_k \le t,\; \exists\, w_1, \dots, w_{k-1} \in \mathcal{V},\ \text{s.t.}:\ u \overset{r_1}{\sim} w_1 \overset{r_2}{\sim} \cdots \overset{r_{k-1}}{\sim} w_{k-1} \overset{r_k}{\sim} v \Big\}$$

(nondecreasing) times

space

space-time path



$\vec{d}_{0,0}(o,u) = +\infty$ $\quad$ $\vec{d}_{0,t_1}(o,u) = +\infty$ $\quad$ $\vec{d}_{0,t_2}(o,u) = 5$ $\quad$ $\vec{d}_{0,t_3}(o,u) = 3$

Some Definitions

Def: The **DYNAMICAL BALL** centered at o, with radius d in $(G_t)_{t\geq 0}$ is the following dynamical rooted graph :

$$\left(\mathcal{B}_t^d(G), o\right)_{t\geq 0} = \left(\left(V_t^d(G), E_t^d(G)\right), o\right)_{t\geq 0}$$

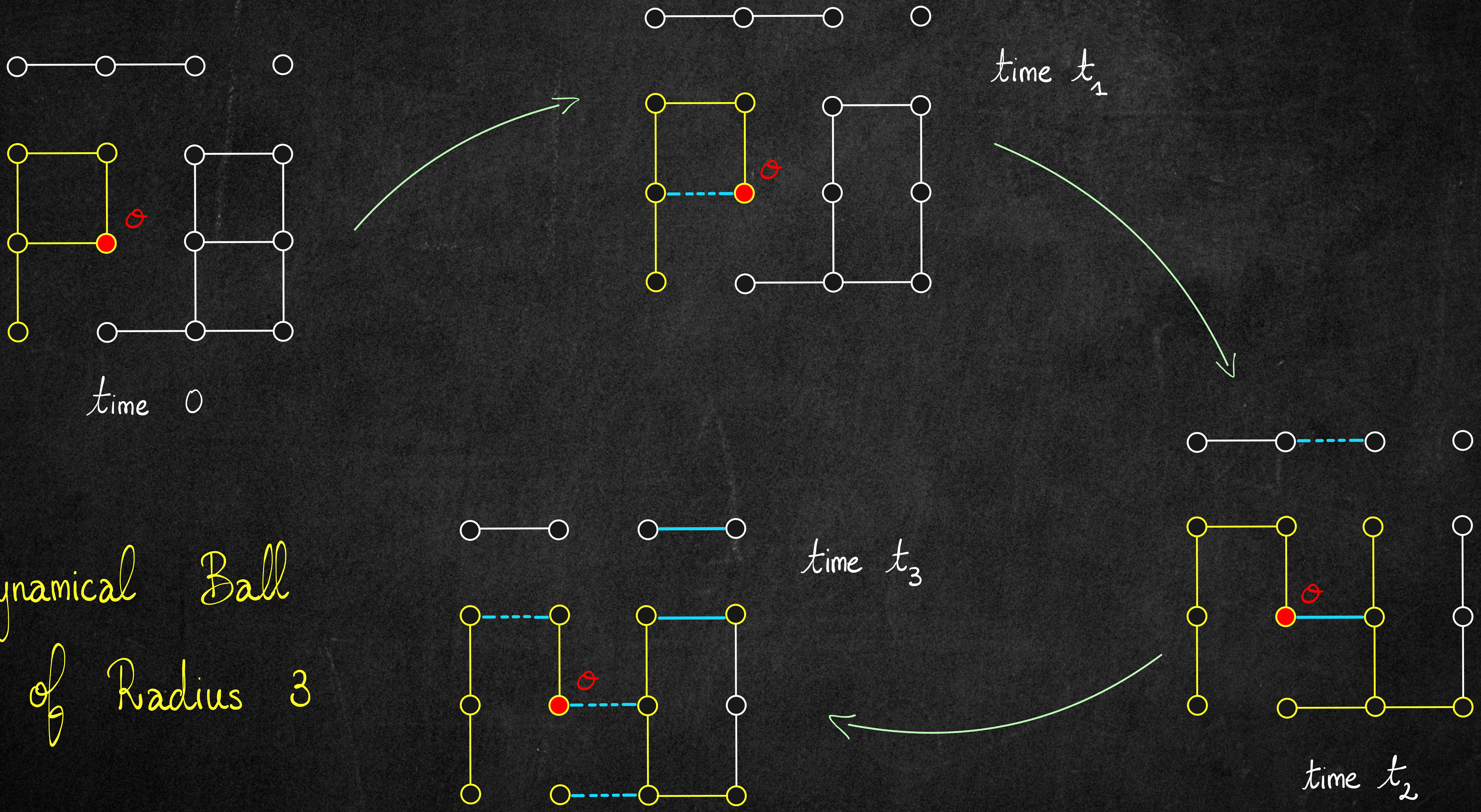
where for all time $t \geq 0$:

$$V_t^d(G) = \left\{ u \in V_t : \vec{d}_{0,t}(o,u) \leq d \right\}$$

and

$$E_t^d(G) = \left\{ u \overset{t}{\sim} v : u, v \in V_t^d(G) \right\}.$$

Some Definitions



Dynamical Ball of Radius 3

Local Weak Convergence

Def: We say that a sequence $\{ (G_t^n, o_n)_{t>0}, n \geq 1 \}$ of dynamical rooted graphs **CONVERGES LOCALLY** to a dynamical rooted graph $(G_t^\infty, o)_{t\geq 0}$, if

$$\forall d \geq 0, \ \exists N_d \geq 1, \ \forall n \geq N_d \ . \quad \left(B_t^d(G^n), o_n\right)_{t\geq 0} \equiv \left(B_t^d(G^\infty), o\right)_{t \geq 0} .$$

isomorphism

Proposition: It exists a distance d_{loc} which metrizes this notion of local convergence and s.t. $(\mathcal{DG}_\bullet, d_{loc})$ is a complete metric space.

Local Weak Convergence

with n vertices

Def: We say that a sequence $\{(G_t^n)_{t>0}, n \geq 1\}$ of finite (random) dynamical graphs **CONVERGES LOCALLY WEAKLY** to a random dynamical rooted graph $(G_t^\infty, o)_{t\geq 0}$, if for all bounded and continuous function $h : \mathcal{DG}_\bullet \longrightarrow \mathbb{R}$:

$$\mathbb{E}\left[\frac{1}{|V_n|}\sum_{v\in V_n} h\left((G_t^n, o_n)_{t>0}\right)\right] \xrightarrow[n\to+\infty]{} \mathbb{E}\left[h\left((G_t^\infty, o)_{t\geq 0}\right)\right].$$

Equivalently,

$$(G_t^n, o_n)_{t>0} \xrightarrow[n\to+\infty]{\text{Law}} (G_t^\infty, o)_{t\geq 0},$$

where o_n is a uniform vertex of V_n.

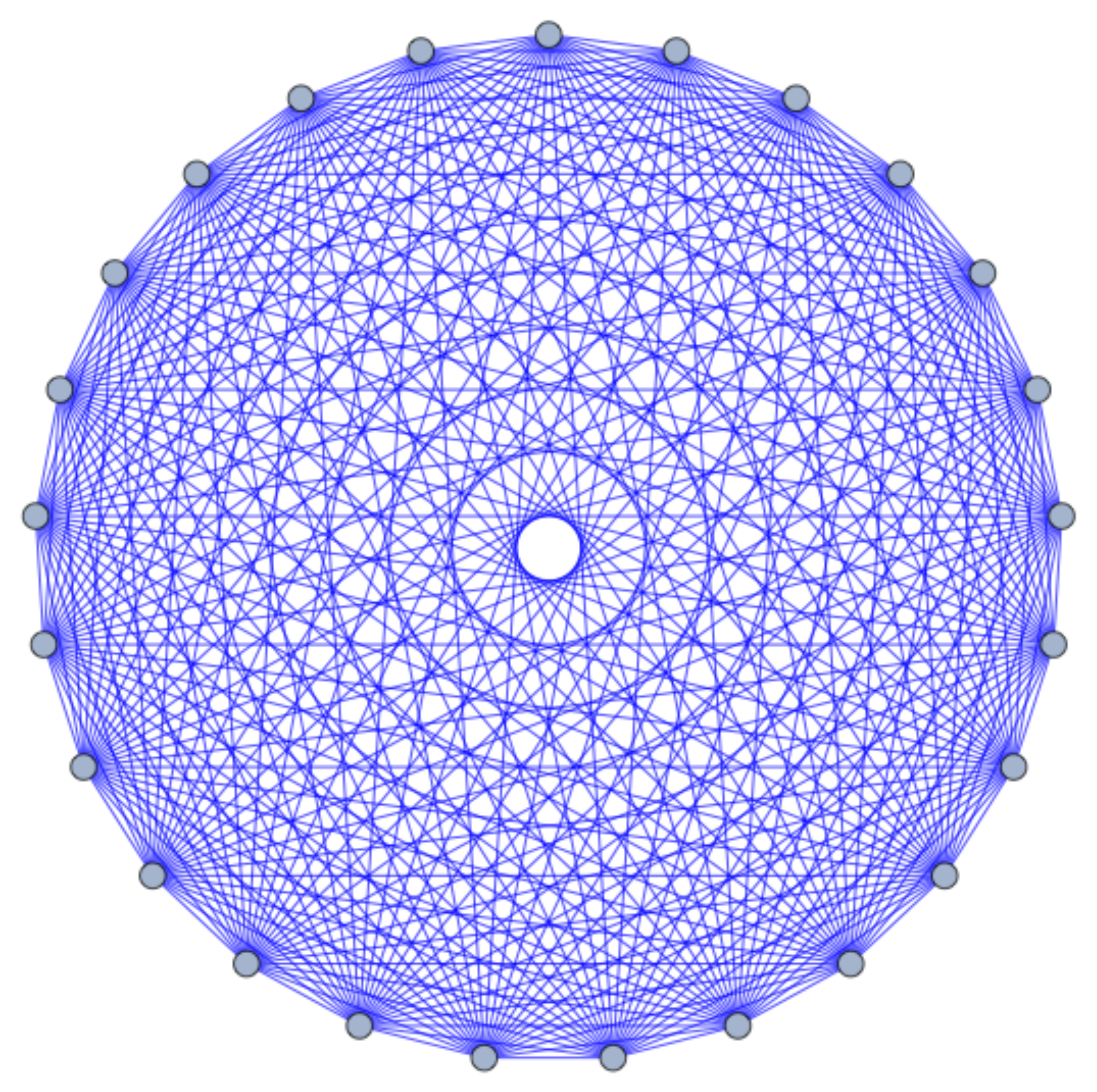
DYNAMICAL ERDÖS-RÉNYI RANDOM GRAPH & MAIN RESULTS

- The Model & The Limit
- Main Results

The Model

Consider the complete graph $K_n = (V_n, E_n)$, where

$V_n = \{1, 2, \dots, n\}$ and $E_n = \{u \sim v : u \neq v\}$.



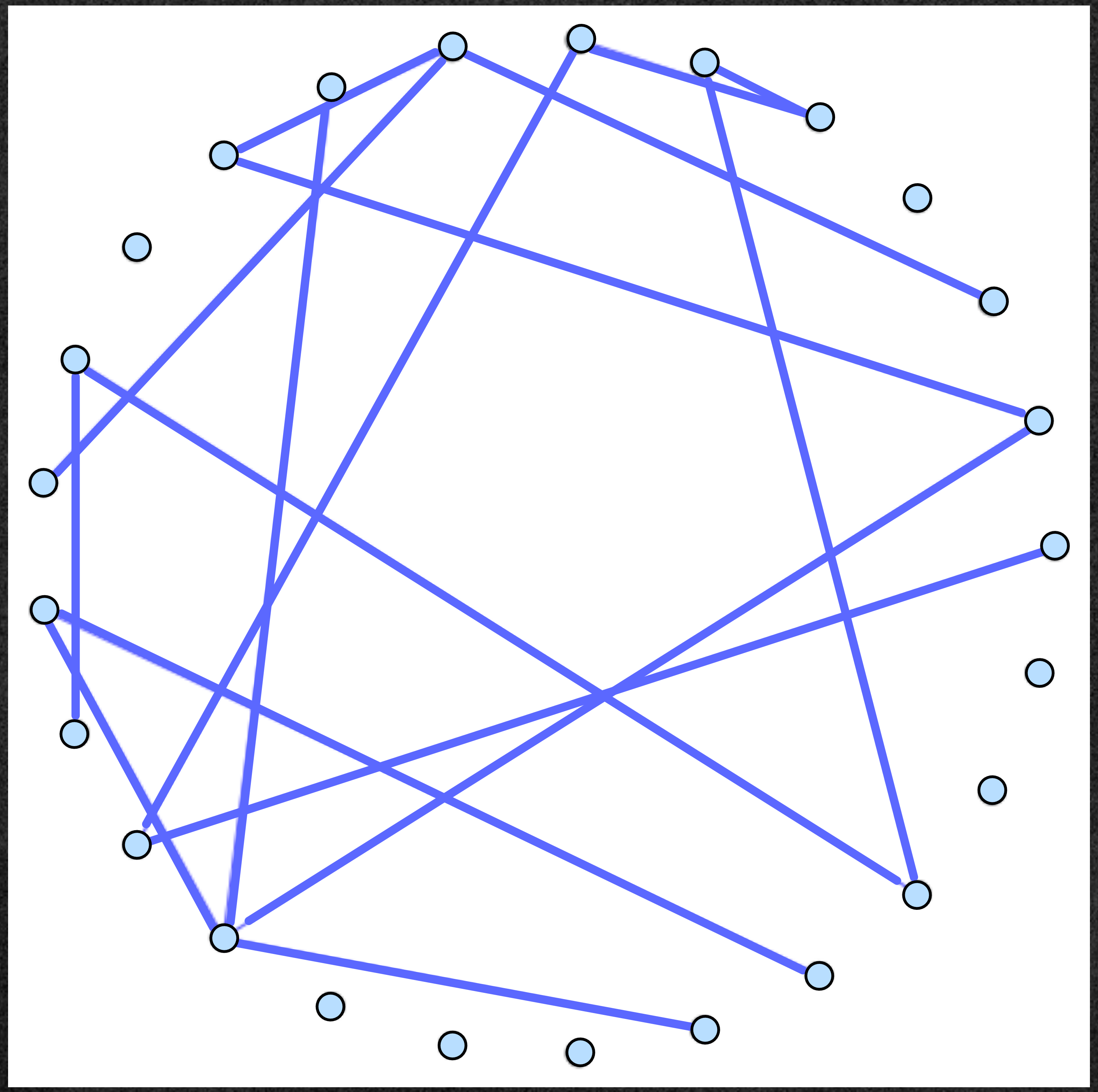
K_{25}

Dynamic : **Dynamical Percolation** Häggström - Peres & Steif, 1997

The Model

Dynamic: Dynamical Percolation   Häggström - Peres & Steif, 1997

⇝ at time 0 : $\mathcal{G}_0^{(n),\kappa,\beta}$ is an Erdős - Rényi random graph with parameter $\frac{\kappa}{n}$



The Model

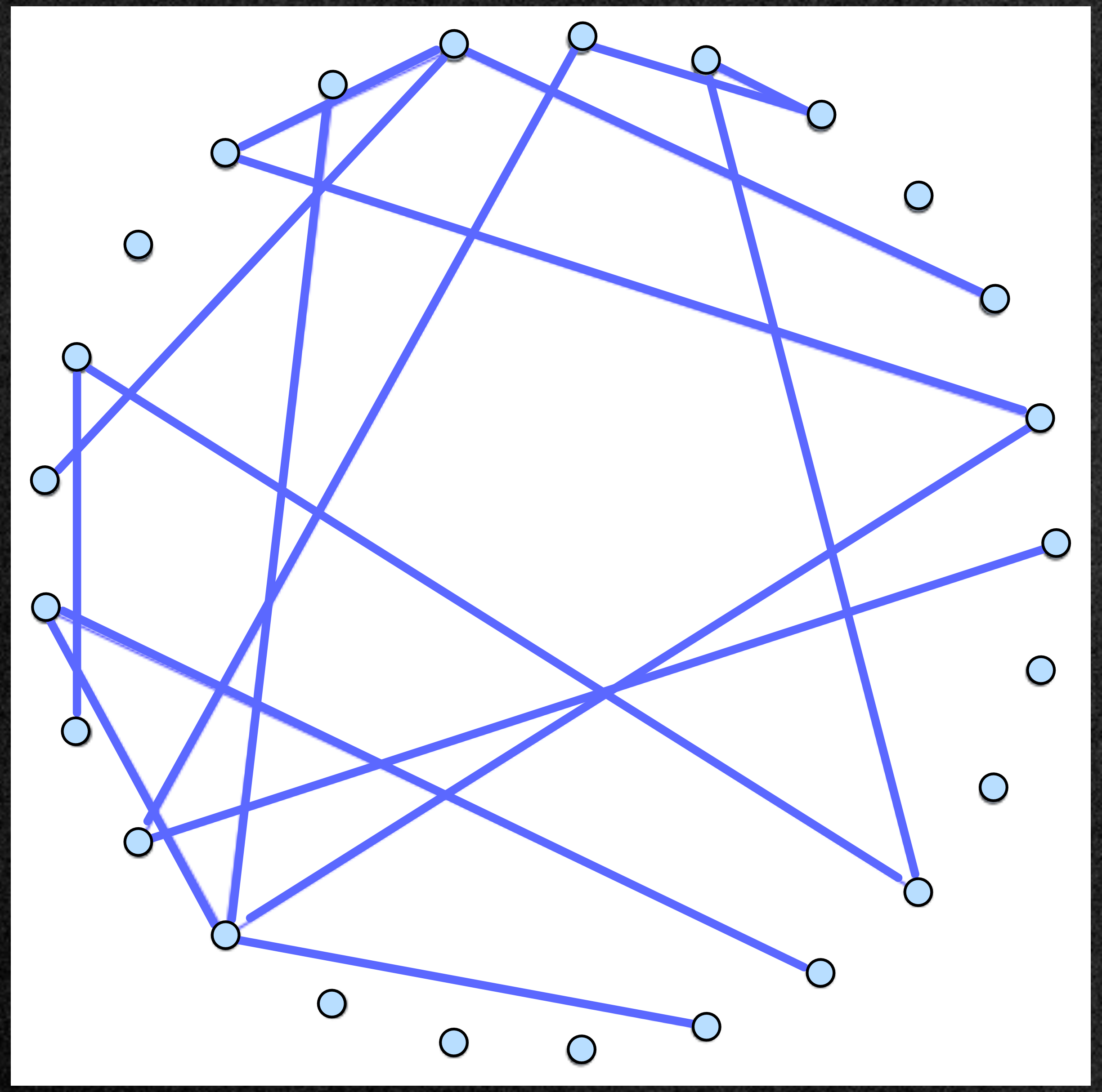
Dynamic : **Dynamical Percolation** Häggström - Peres & Steif , 1997

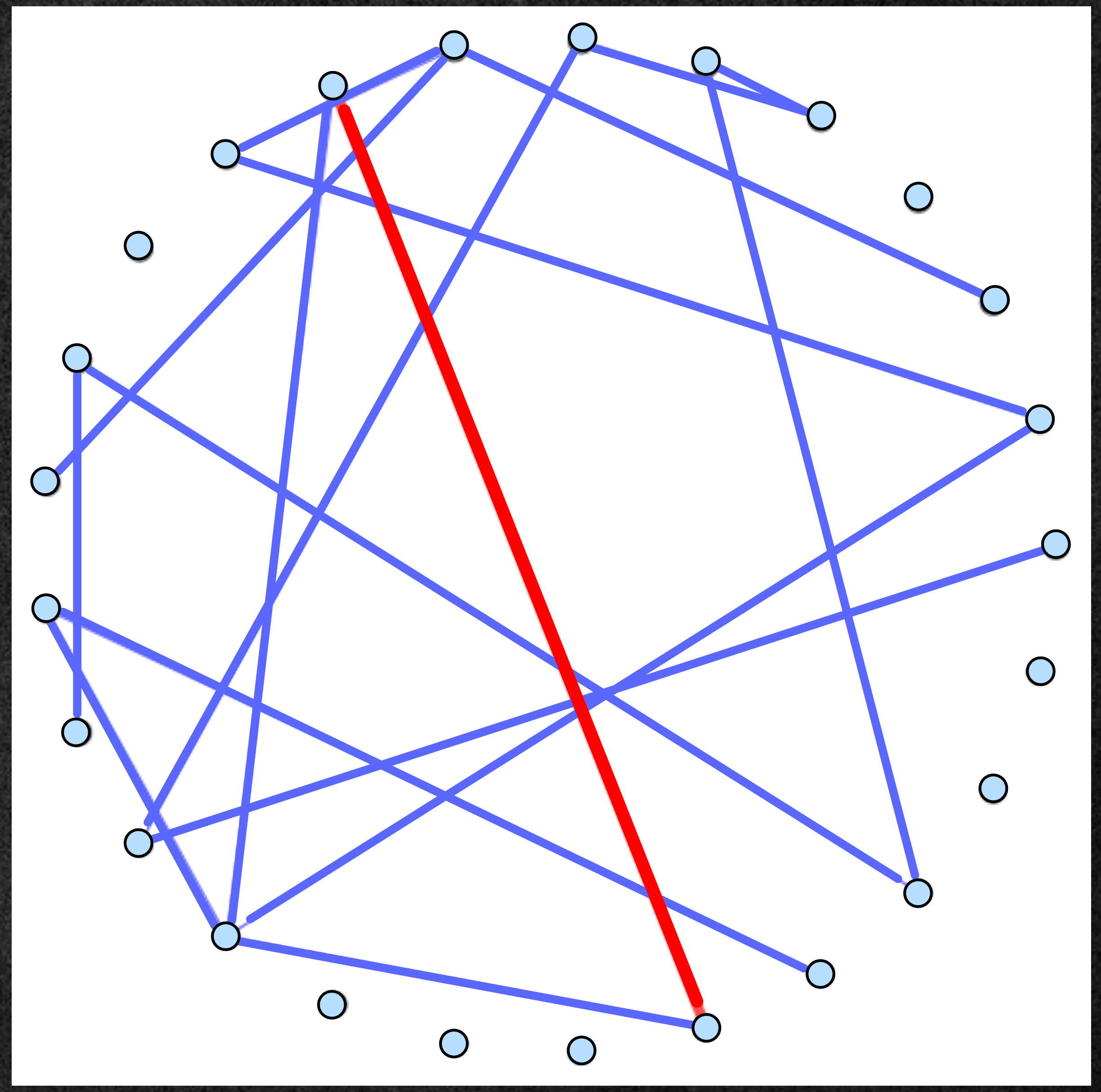
⇝ at time 0 : $\mathcal{G}_0^{(n),\kappa,\beta}$ is an Erdös - Rényi random graph with parameter $\frac{\kappa}{n}$

⇝ a closed edge becomes open at rate $\beta \frac{\kappa}{n}$

updating parameter (β)

connection parameter (κ)





The Model

Dynamic : Dynamical Percolation Häggström - Peres & Steif , 1997

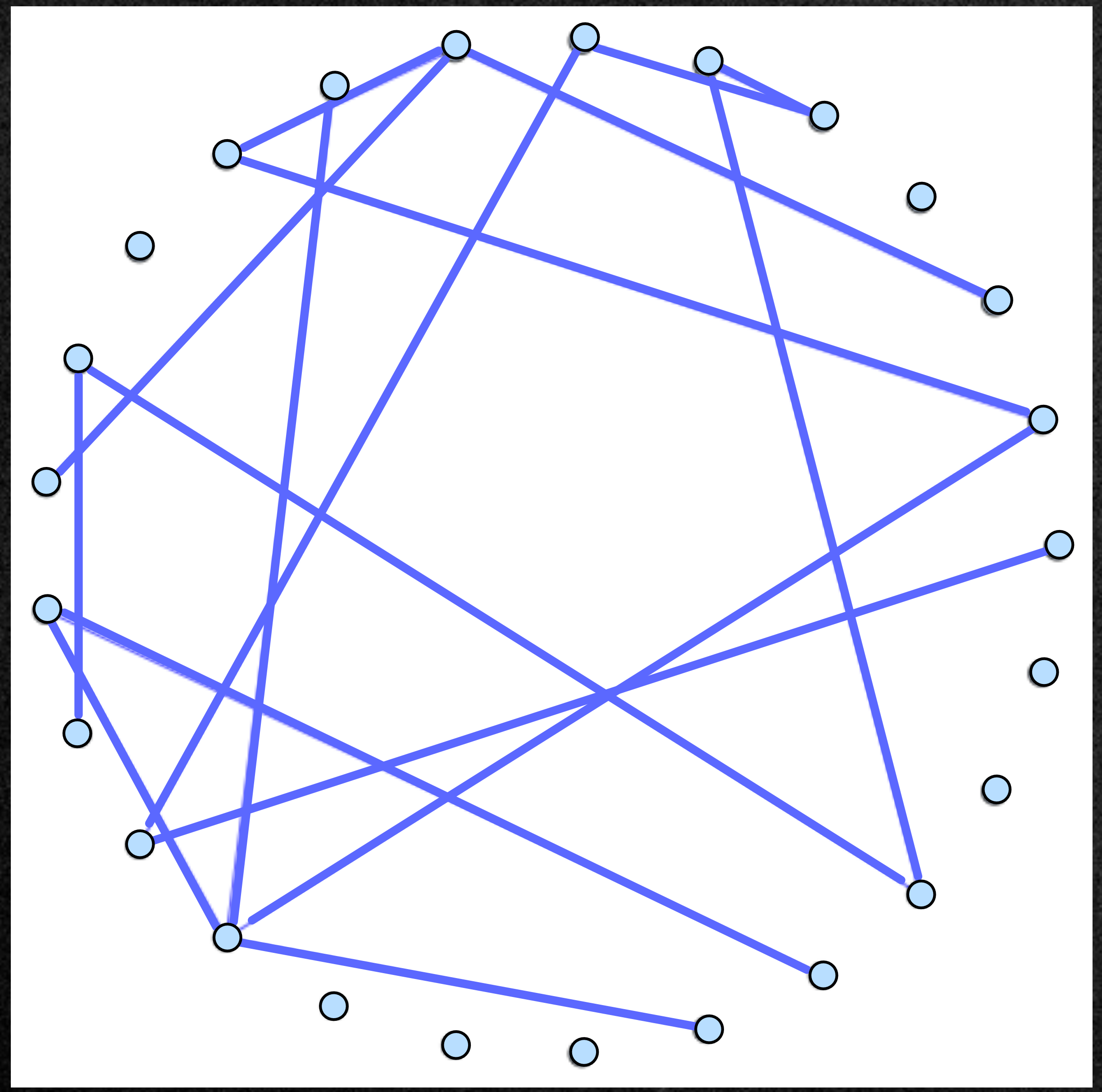
⇝ at time 0 : $\mathcal{G}_0^{(n),\kappa,\beta}$ is an Erdös - Rényi random graph with parameter $\frac{\kappa}{n}$

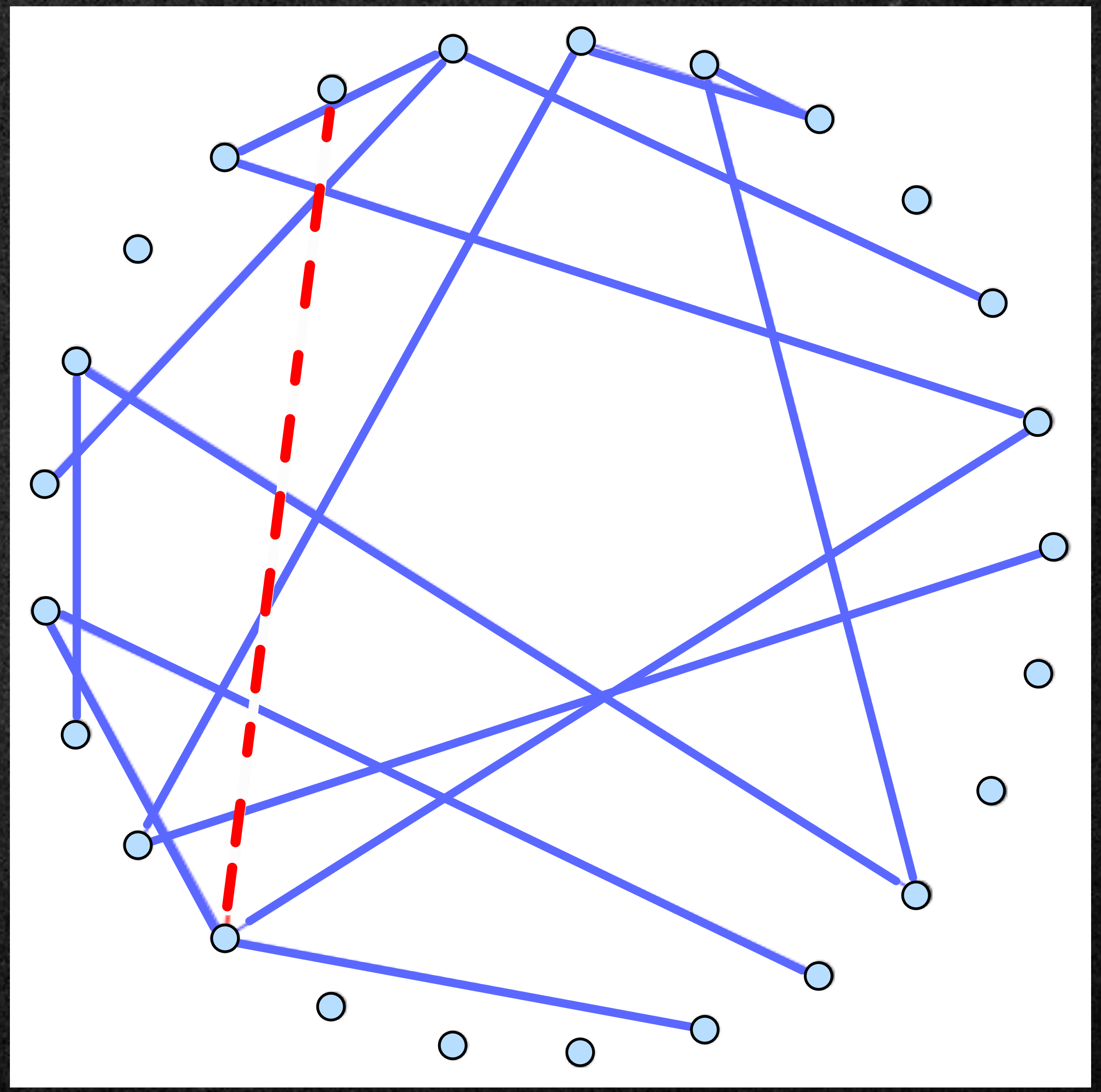
⇝ a closed edge becomes open at rate $\beta\frac{\kappa}{n}$

updating parameter

connection parameter

⇝ an open edge becomes closed at rate $\beta\cdot\left(1-\frac{\kappa}{n}\right)$





The Model

3 important Remarks:

- It is a Markov process.

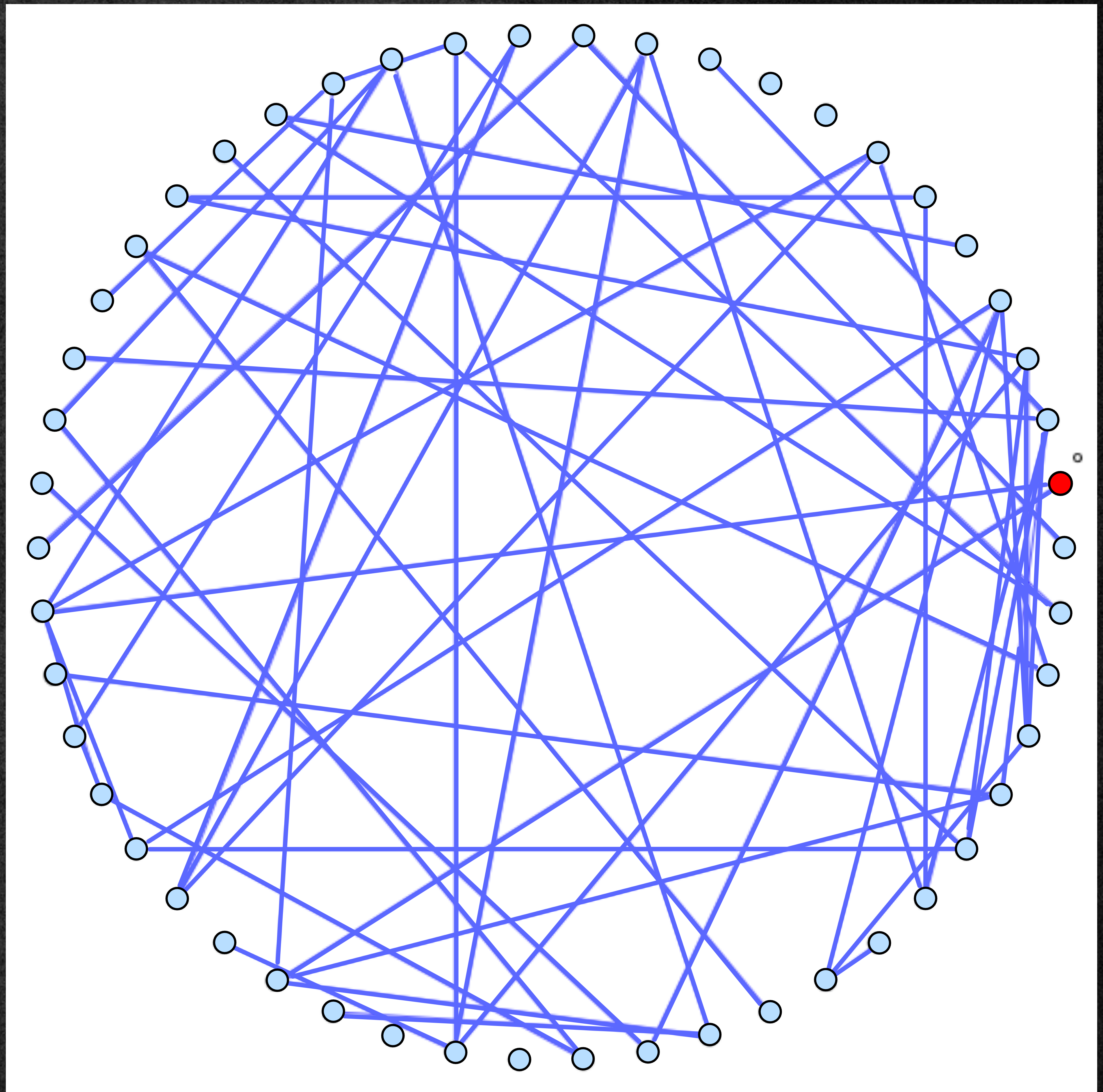
- It is a stationary dynamic:

  at each time $t \geq 0$, the law of the graph $\mathcal{G}_t^{(n),\kappa,\beta}$ is the law of $ER(n, \kappa/n)$.

- $\mathbb{P}\left[\text{two edges flip simultaneously}\right] = 0$.

How does look like the Dynamical Ball ?

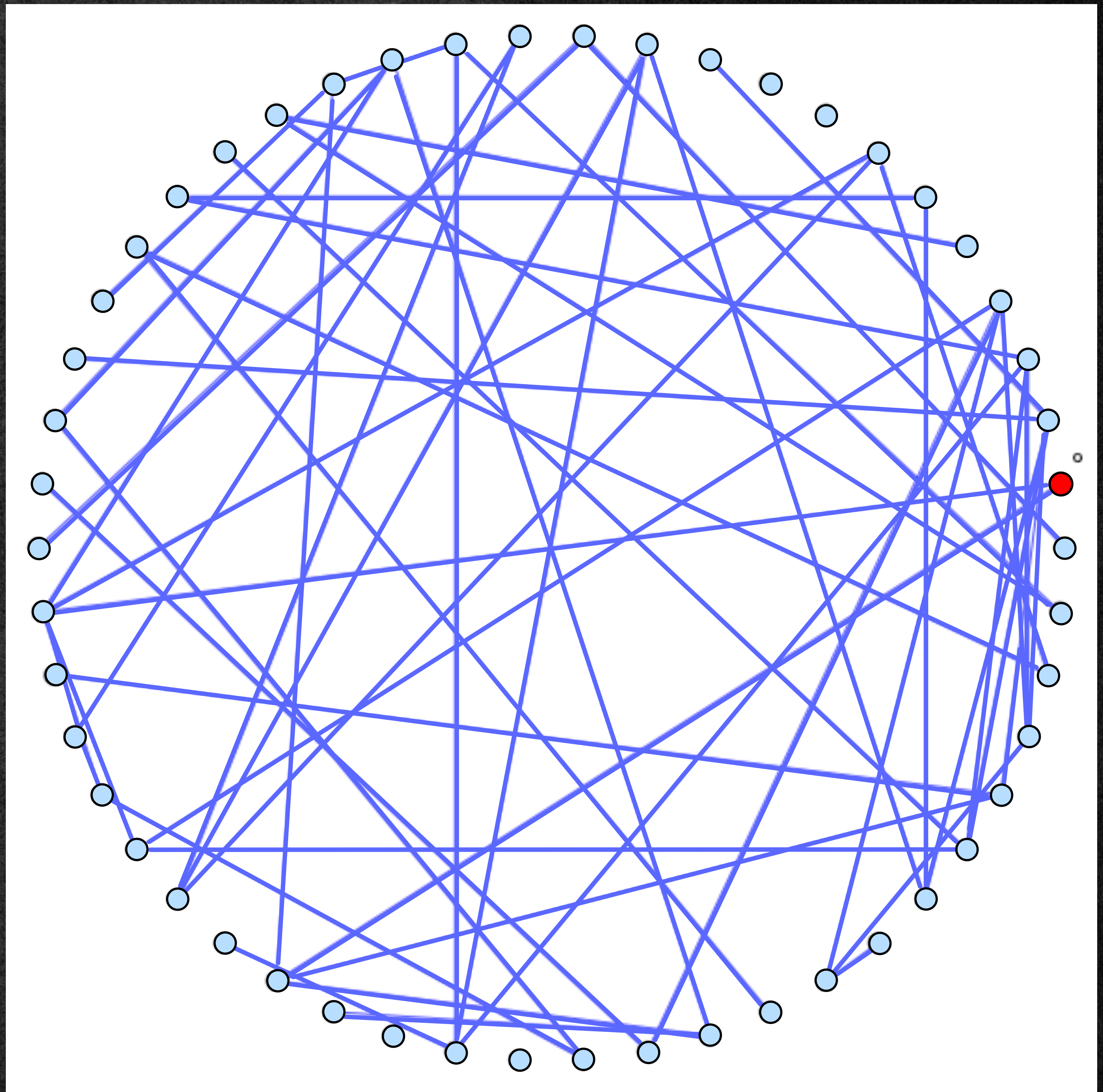
Fix a root $o \in V_n$, and a distance $d \geq 1$.



How does look like the Dynamical Ball ?

Fix a root $o \in V_n$, and a distance $d \geq 1$.

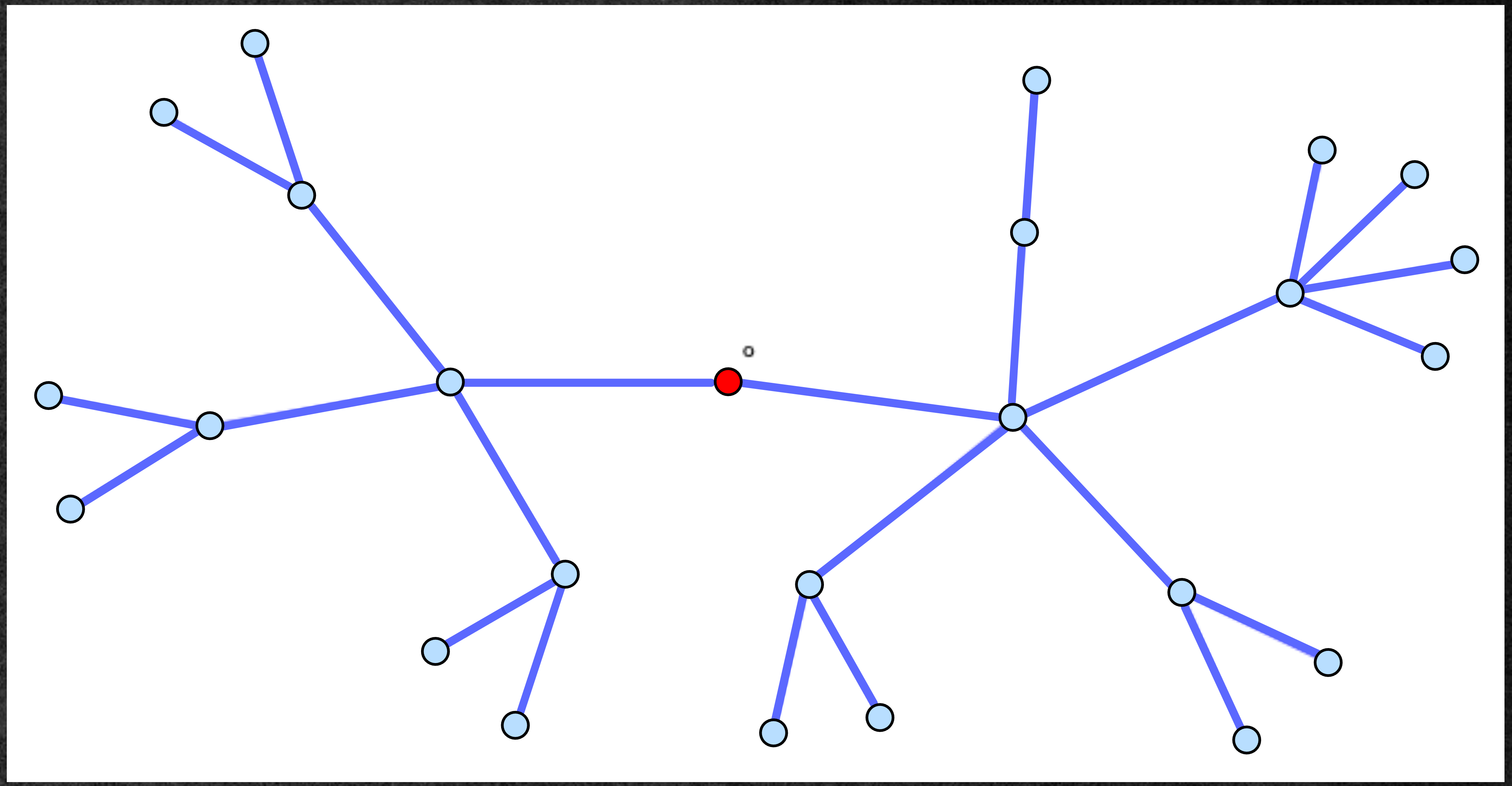
At time $t = 0$, the dynamical ball is exactly the ball in an ER($\kappa, \kappa/n$).



How does look like the Dynamical Ball ?

Fix a root $o \in V_n$, and a distance $d \geq 1$.

$\approx$ PGW(κ) tree



At time $t = 0$, the dynamical ball is exactly the ball in an $ER(\kappa, \kappa/n)$.

⇝ When n is large enough, w.h.p. this is the ball of radius d of a PGW(κ) tree.

with high probability

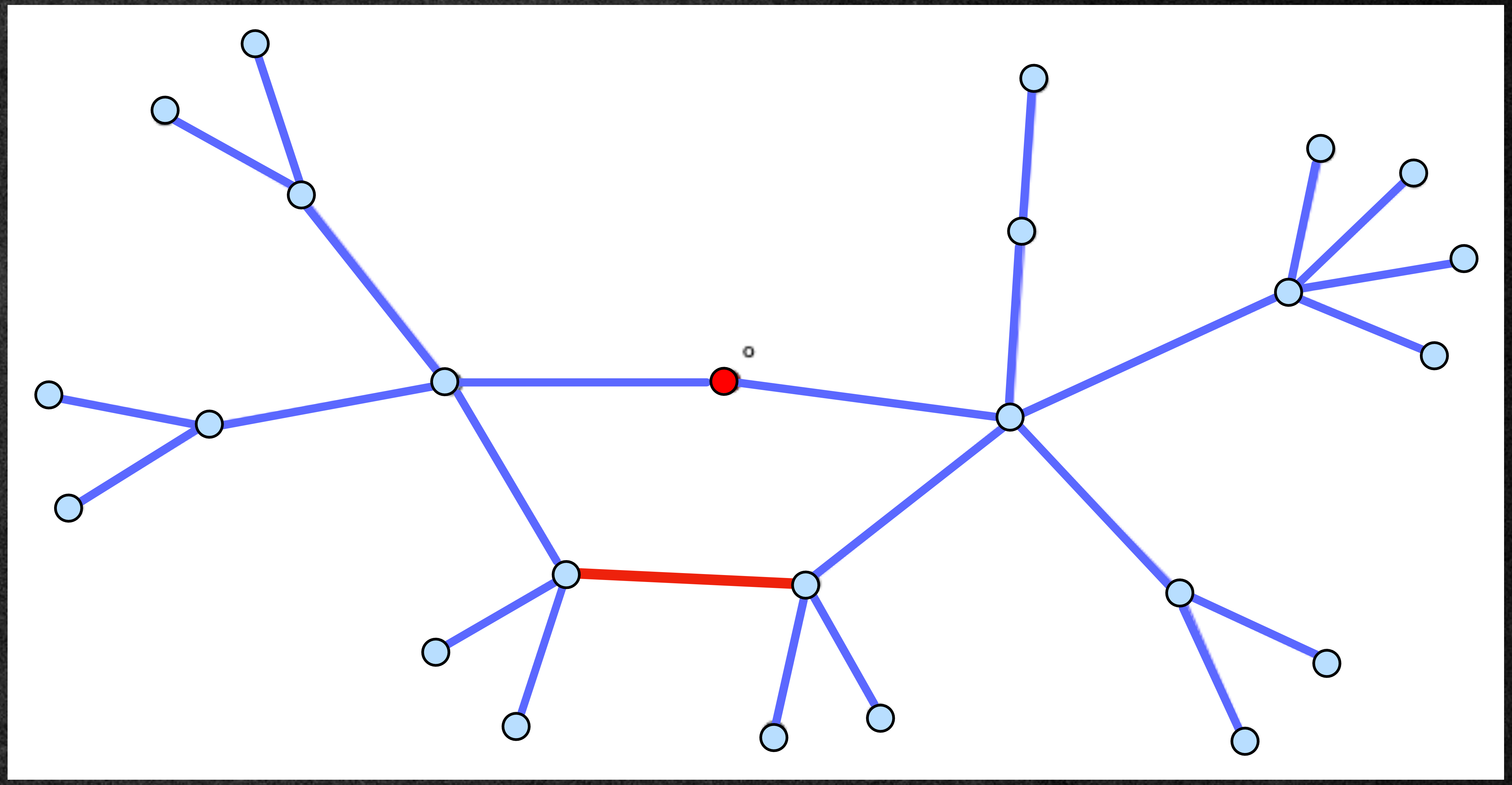
Then "three" transitions may occur.

How does look like the Dynamical Ball ?

1. An edge inside the ball appears.

This happens at rate

$$\leq \beta \cdot \frac{K}{n} \cdot \#\{\text{vertices at } t=0\}^2 \xrightarrow[n\to+\infty]{(\mathbb{P})} 0.$$



⇝ "This never occurs in the limit."

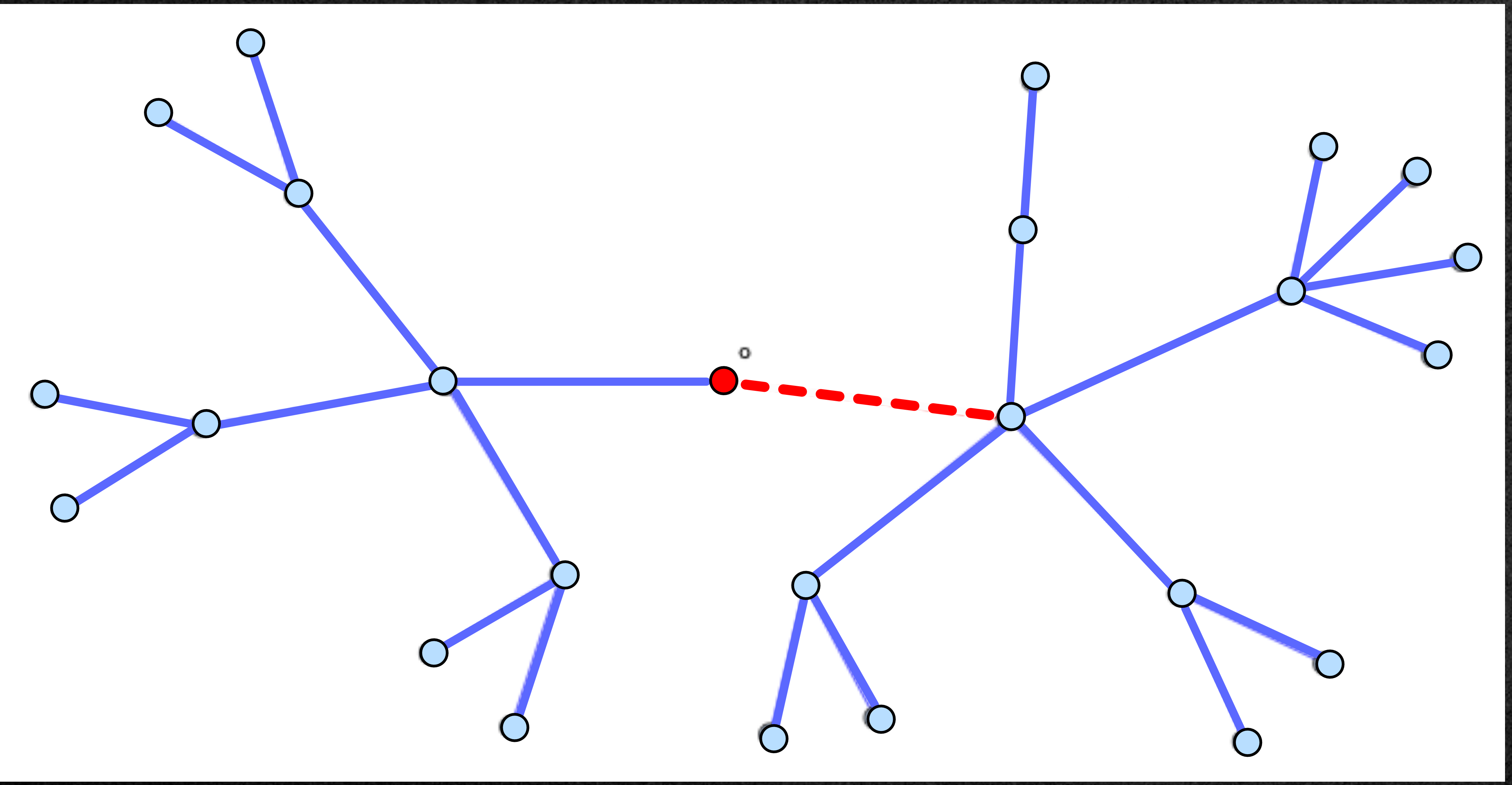
How does look like the Dynamical Ball ?

1. An edge inside the ball appears.

This happens at rate

$$\leq \beta \cdot \frac{K}{n} \cdot \#\{\text{vertices at } t=0\}^2 \xrightarrow[n \to +\infty]{(\mathbb{P})} 0 .$$

⇝ " This never occurs in the limit. "



2. An edge inside the ball disappears.

This happens at rate

$$\beta \cdot \left(1 - \frac{K}{n}\right) \xrightarrow[n \to +\infty]{} \beta .$$

⇝ " This does not break the tree / forest structure. "

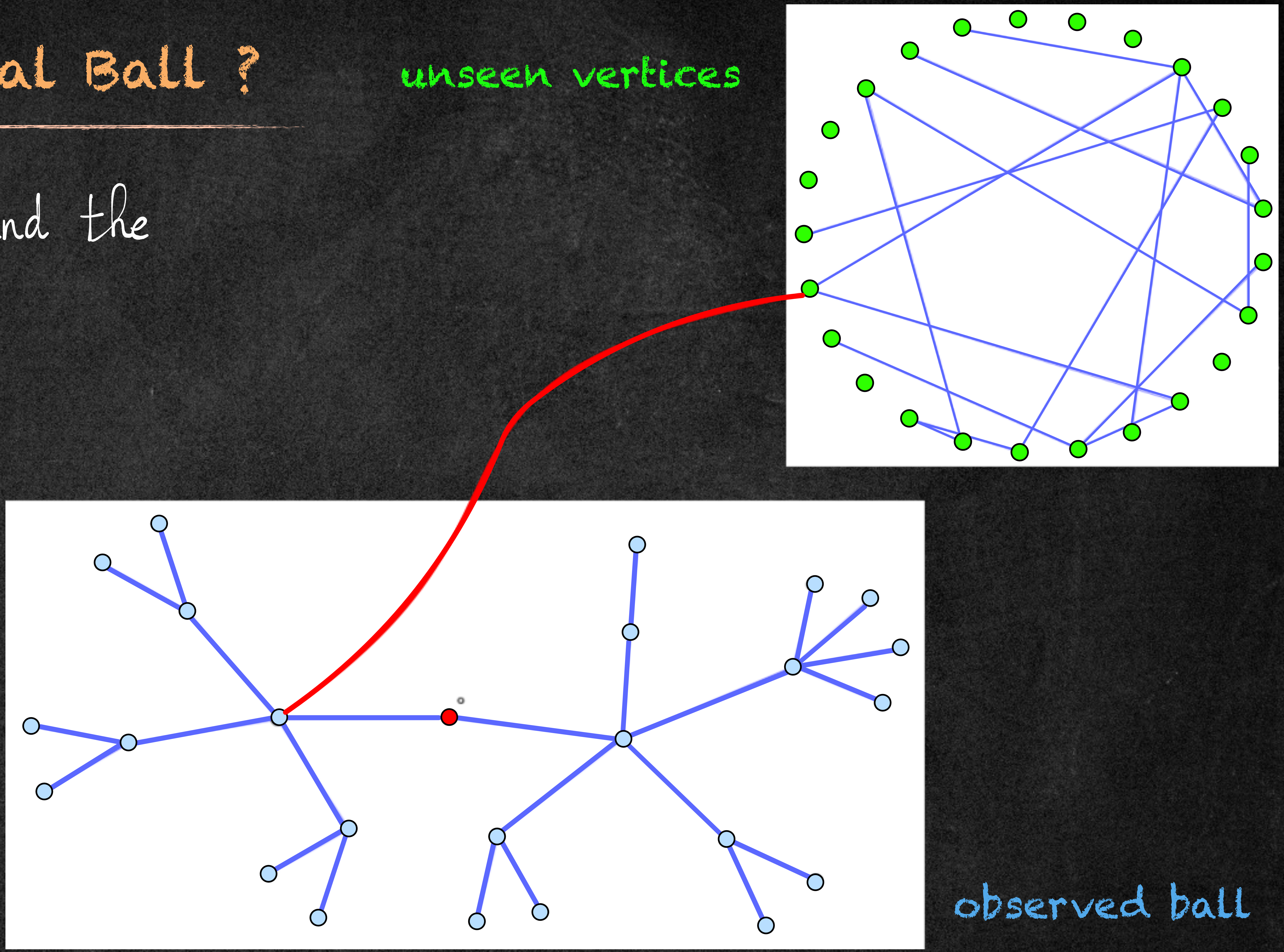
How does look like the Dynamical Ball ?

3. An edge between the observed ball and the remaining unseen vertices appears.

This happens at rate

$$\beta \cdot K - \beta \cdot \frac{K}{n} \cdot \#\{\text{vertices at } t=0\}$$

$$\xrightarrow[n \to +\infty]{(\mathbb{P})} \beta \cdot K .$$



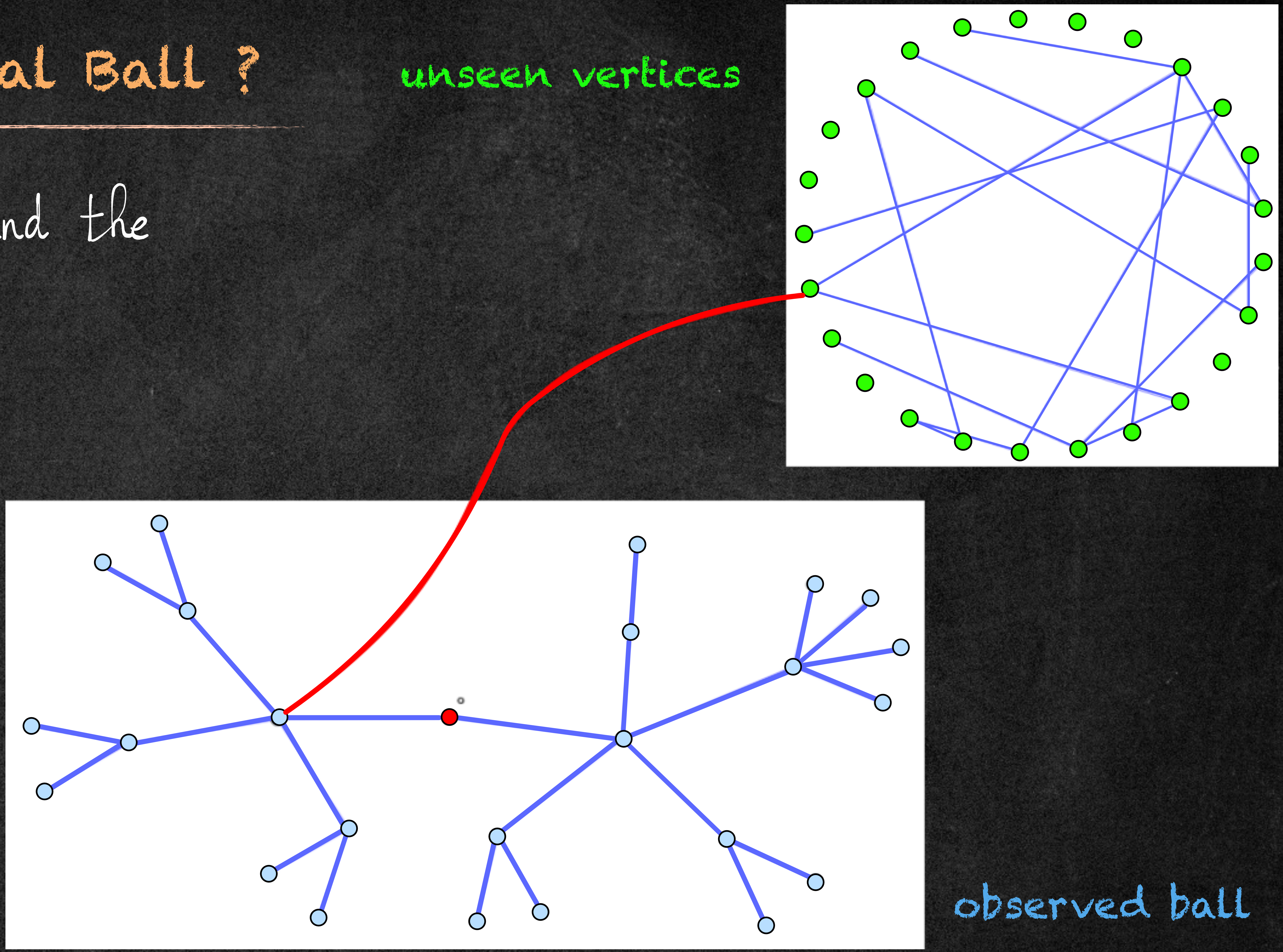
How does look like the Dynamical Ball ?

3. An edge between the observed ball and the remaining unseen vertices appears.

This happens at rate

$$\beta \cdot K - \beta \cdot \frac{K}{n} \cdot \# \{ \text{vertices at } t = 0 \}$$

$$\xrightarrow[n \to +\infty]{(\mathbb{P})} \beta \cdot K .$$



What is the component added ?

How does look like the Dynamical Ball ?

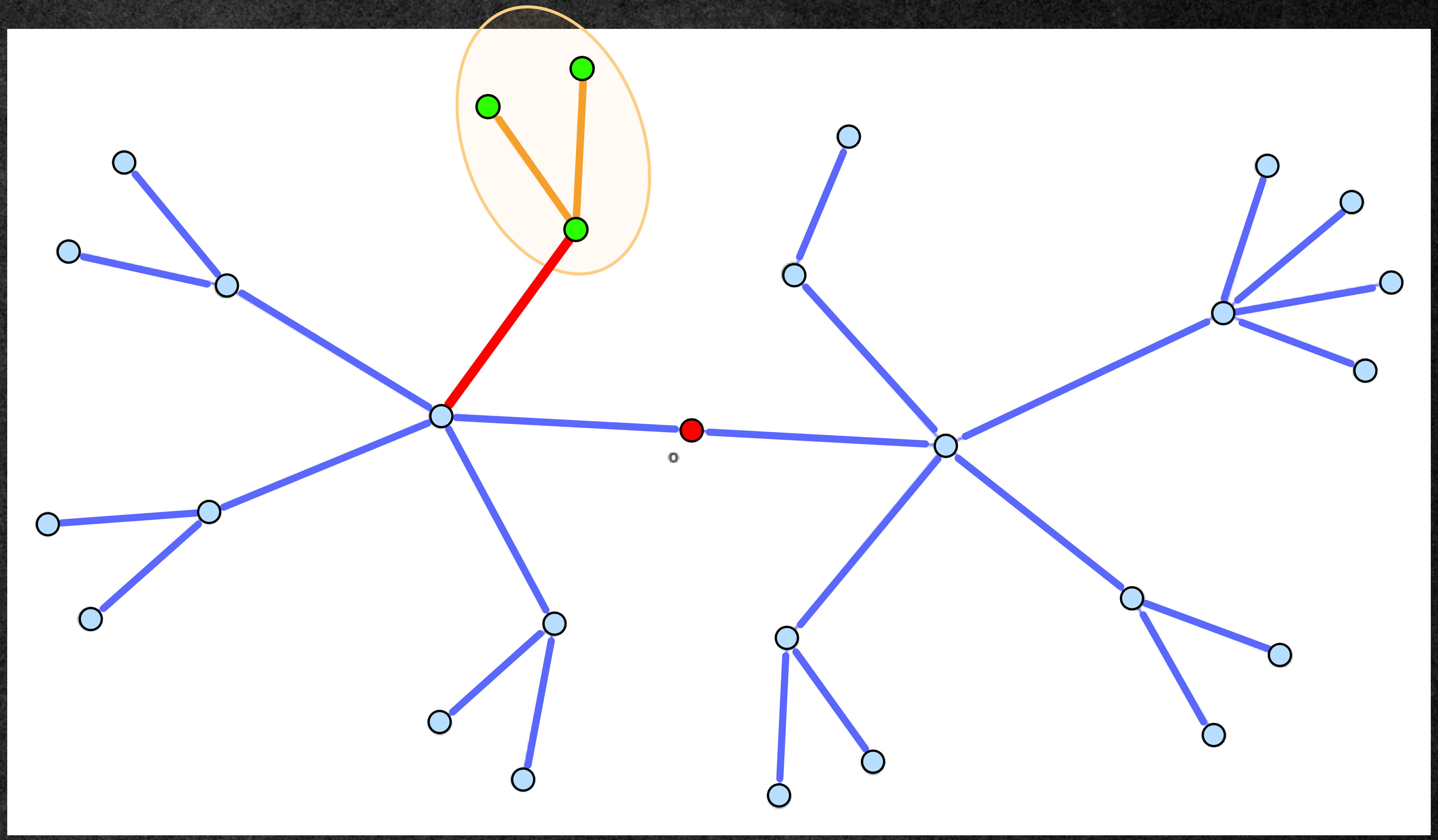
3. An edge between the observed ball and the remaining unseen vertices appears.

This happens at rate

$$\beta \cdot K - \beta \cdot \frac{K}{n} \cdot \#\{\text{vertices at } t=0\}$$

$$\xrightarrow[n \to +\infty]{(\mathbb{P})} \beta \cdot K .$$

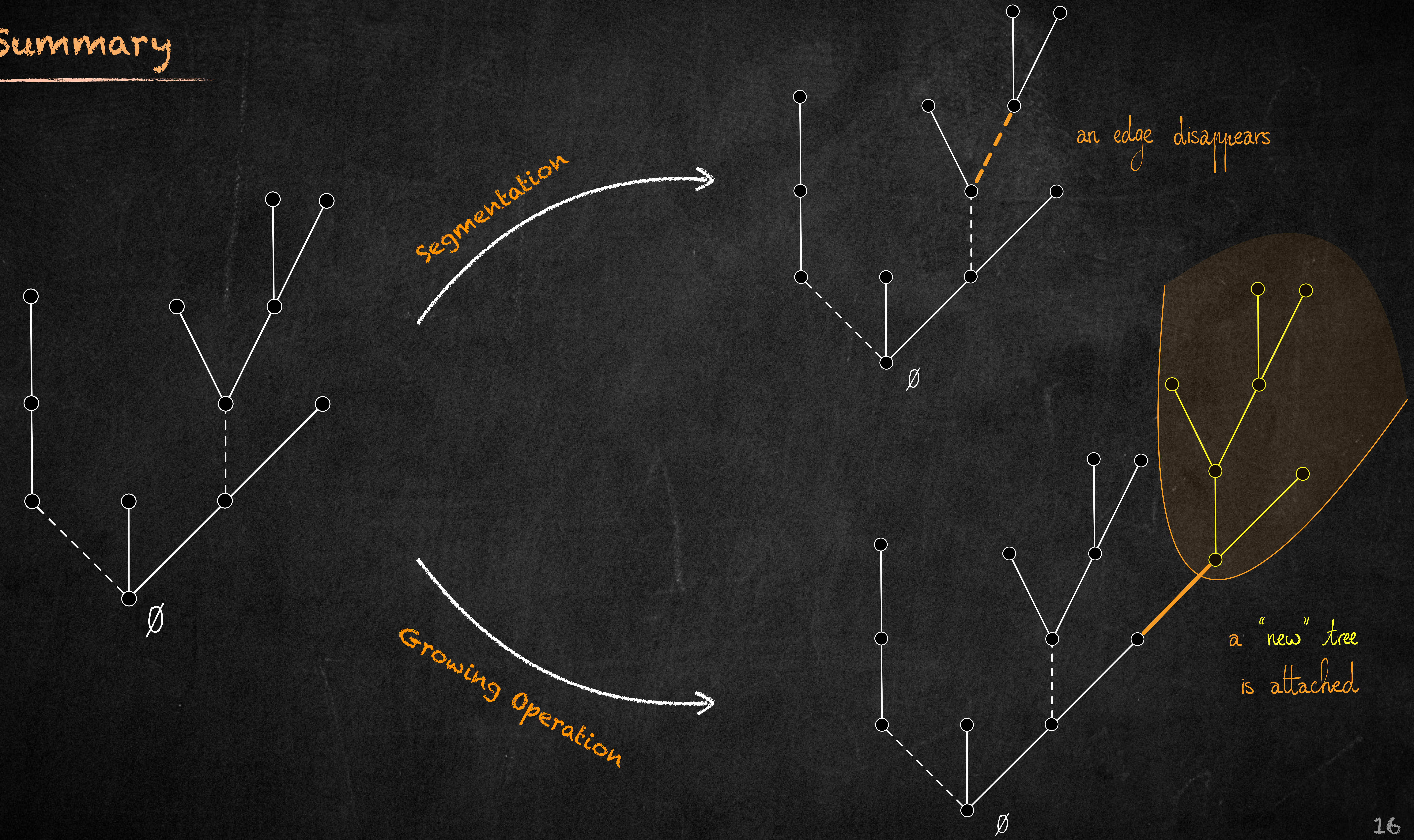
a ball of a PGW(K) TREE is attached



Question: What is the component added ?

- the evolutions of the ball and the graph formed by the other vertices are independent (independence of the evolution of the edges)
- the graph formed by the vertices outside the ball is an ER (stationarity property)

In Summary



Segmentation
Growing Operation
Ø
Ø
Ø
an edge disappears
a "new" tree
is attached

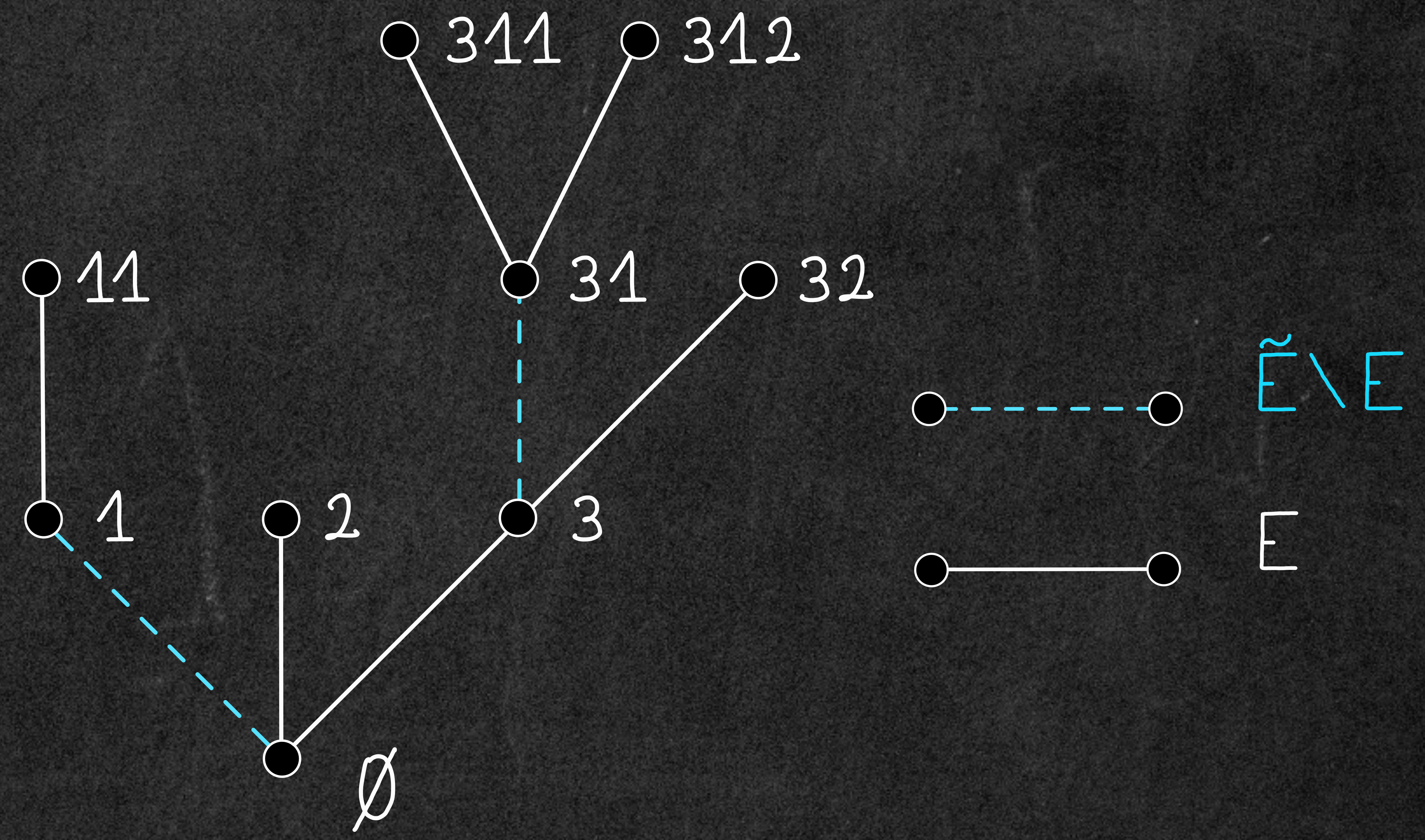
Segmented Trees

Def: An **(ordered) SEGMENTED TREE** is a triplet $(V, \tilde{E}, E)$, where

- $(V, \tilde{E})$ is an ordered tree ; and
- E is a subset of $\tilde{E}$.

An ordered segmented tree.

Segmented edges are dashed.



Notations: $\mathbb{ST} = \{ \text{segmented tree} \}$

$\mathbb{ST}^d = \{ \text{segmented tree with height} \leq d \}$.

The Growth-and-Segmentation PGW Tree

Def: A process $(\mathcal{T}_t^{\kappa,\beta})_{t\geq 0}$ on $\mathbb{ST}$ is a **GROWTH-and-SEGMENTATION PGW TREE** (GSPGW(κ,β)) with parameters $\kappa, \beta \geq 0$, if

- at time 0, $\mathcal{T}_0^{\kappa,\beta}$ is a PGW(κ) tree ;
- a present edge in $\mathcal{T}_t^{\kappa,\beta}$ is segmented at rate β ; and
- for each vertex $u \in \mathcal{T}_t^{\kappa,\beta}$, a PGW($\kappa$) tree is attached to u at rate $\beta \cdot \kappa$.

Theorem: $(\mathcal{T}_t^{\kappa,\beta})_{t\geq 0}$ is a Markov process on $\mathbb{ST}$ with càdlàg paths with probability one.

Main Results

Theorem D. , Jacob - 2022+

$$\mathrm{DER}(\kappa,\beta) \xrightarrow[n \longrightarrow +\infty]{\text{loc}} \mathrm{GSPGW}(\kappa,\beta)$$

Main Results

Theorem D., Jacob - 2022+

$$\mathrm{DER}(\kappa,\beta) \xrightarrow[n \longrightarrow +\infty]{\text{loc}} \mathrm{GSPGW}(\kappa,\beta)$$

Coupling Method

Theorem D., Jacob - 2022+

times $\underset{n\to+\infty}{\longrightarrow} +\infty$

distances $\underset{n\to+\infty}{\longrightarrow} +\infty$

Assume that $\left(\kappa\left(1+\beta t_n\right)\right)^{d_n+2} = o(n)$ as $n \longrightarrow +\infty$, then we can couple $\left(\mathcal{G}_t^{(n),\kappa,\beta}, o\right)_{t\geq 0}$ with $\mathrm{GSPGW}(\kappa,\beta)$ so that:

$$\mathbb{P}\left[\forall s < t_n : \left(\mathcal{B}_s^{d_n}\left(\mathcal{G}^{(n),\kappa,\beta}\right), o\right) = \mathcal{F}_s^{d_n}\right] = 1 - o(1)$$

EXTENSIONS

- Joint LWC Convergence
- Dynamical Inhomogeneous Random Graphs
- Vertex Updating

Some Extensions: Joint LWC

- We also proved joint local limit for the model DER :

$$\left(\left(\mathcal{G}_t^{(n),\kappa,\beta}, \theta_1\right)_{t\geq 0}, \dots, \left(\mathcal{G}_t^{(n),\kappa,\beta}, \theta_k\right)_{t\geq 0} \right) \xrightarrow[n \to +\infty]{\mathcal{L}aw} \left(\left(\mathcal{T}_t^{\circ 1}\right)_{t\geq 0}, \dots, \left(\mathcal{T}_t^{\circ k}\right)_{t\geq 0} \right).$$

↓ k independent copies of the GSPGW(κ,β)

Some Extensions: Dynamical Inhomogeneous Random Graph

- We also proved joint local limit for the model DER .
- We also proved local limit for a more general model :

DYNAMICAL INHOMOGENEOUS RANDOM GRAPH

⇝ at time 0 : Inhomogeneous random graph

Söderberg , 2003
Bollobas - Janson & Riordan , 2007

- Each vertex u has a type $x_u \in S$ Polish metric space endowed with a Borel measure
- Probability of edges : $p_{u,v} := \frac{1}{n} \kappa(x_u, x_v) \wedge 1$

where $\kappa : S \times S \longrightarrow [0, +\infty[$ is a suitable function

Some Extensions: Dynamical Inhomogeneous Random Graph

- We also proved joint local limit for the model DER .

- We also proved local limit for a more general model :

DYNAMICAL INHOMOGENEOUS RANDOM GRAPH

⇝ edges $\{u, v\}$ is refreshed independently from each other at rate

$\beta(x_u, x_v)$, where $\beta : S \times S \longrightarrow [0, +\infty[$ is a suitable function

and is declared open with probability $p_{u,v}$ independently from the past and the other edges.

Some Extensions: Dynamical Inhomogeneous Random Graph

- We also proved joint local limit for the model DER.
- We also proved local limit for a more general model:

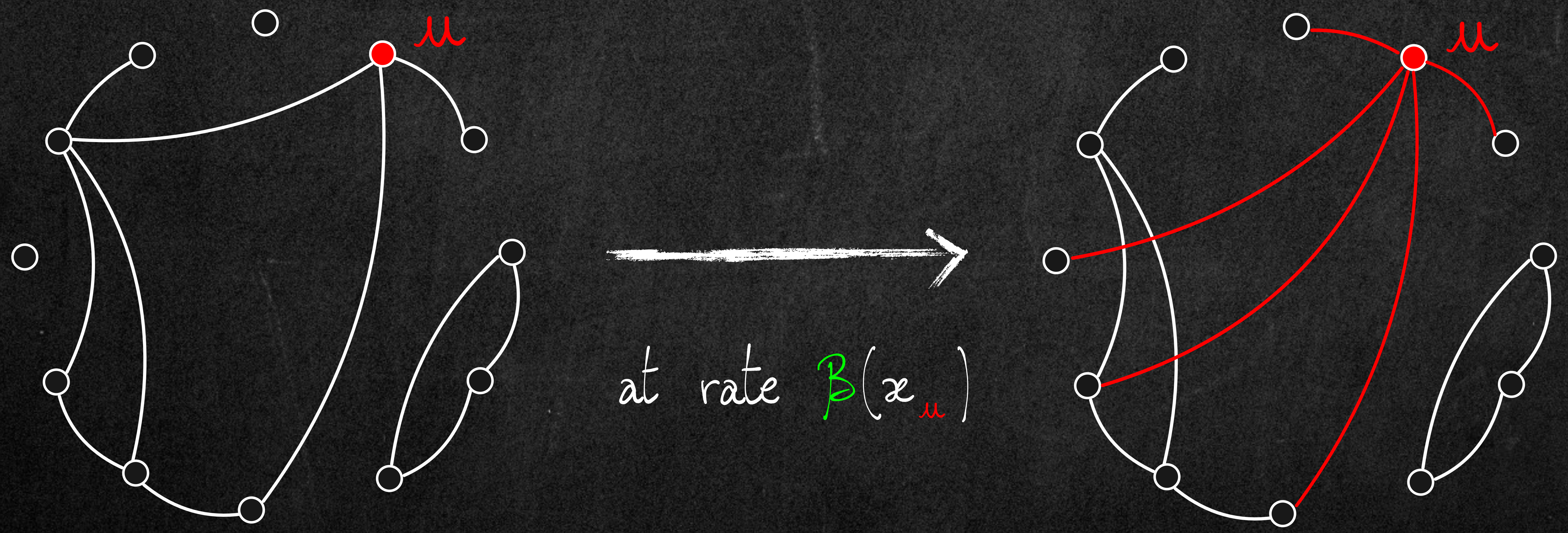
DYNAMICAL INHOMOGENEOUS RANDOM GRAPH

$$\left(\left(\mathcal{G}_t^{(n),K,\beta}, \theta_1 \right)_{t\geq 0}, \ldots, \left(\mathcal{G}_t^{(n),K,\beta}, \theta_k \right)_{t\geq 0} \right) \xrightarrow[n\to+\infty]{\text{Law}} \left(\left(\mathcal{F}_t^{1} \right)_{t\geq 0}, \ldots, \left(\mathcal{F}_t^{k} \right)_{t\geq 0} \right).$$

↓ k independent copies of the Growth-and-Segmentation Multitype PGW Tree

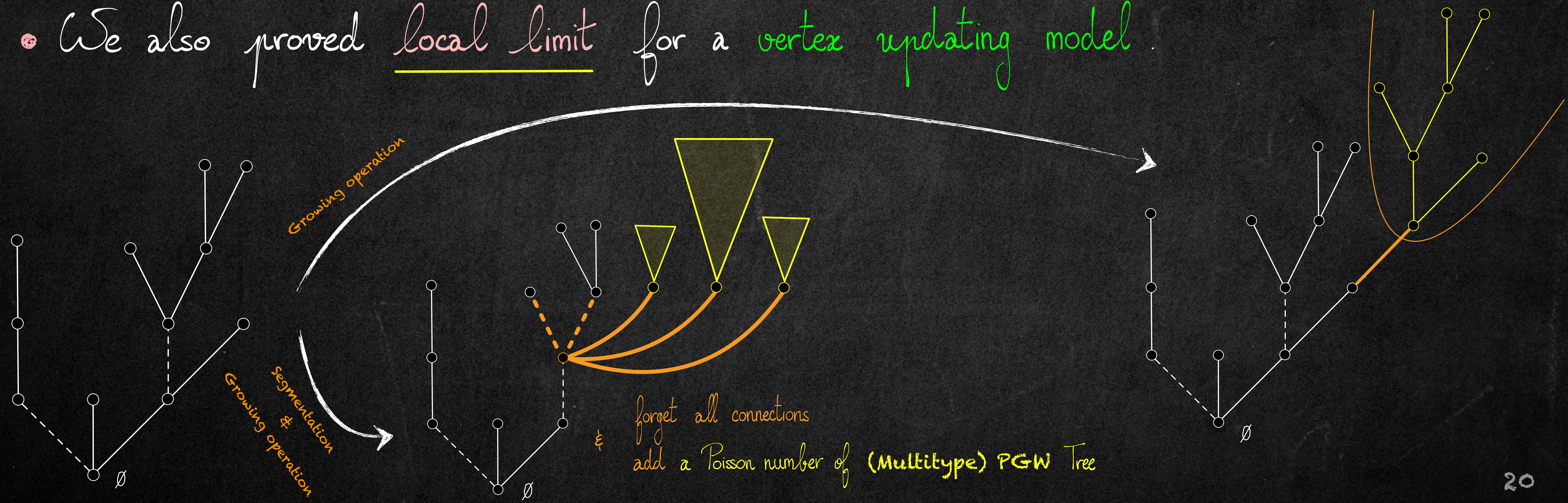
Some Extensions: Vertex Updating

- We also proved joint local limit for the model DER.
- We also proved local limit for a more general model: DIRG.
- We also proved local limit for a vertex updating model.



Some Extensions: Vertex Updating

- We also proved joint local limit for the model **DER** .
- We also proved local limit for a more general model : **DIRG** .
- We also proved local limit for a vertex updating model .



THANK YOU !

Do you have any question ?

The End !

RELATED QUESTIONS ?

- Conclusion

Other Questions ?

- Find another limit for another model of dynamical graphs

⇝ Configuration model — Avena - Güldaş - van der Hofstad & den Hollander, 2018

Work in progress

- Develop the theory of **LCW** for dynamical graphs

⇝ Characterization , Unimodularity , Spectrum

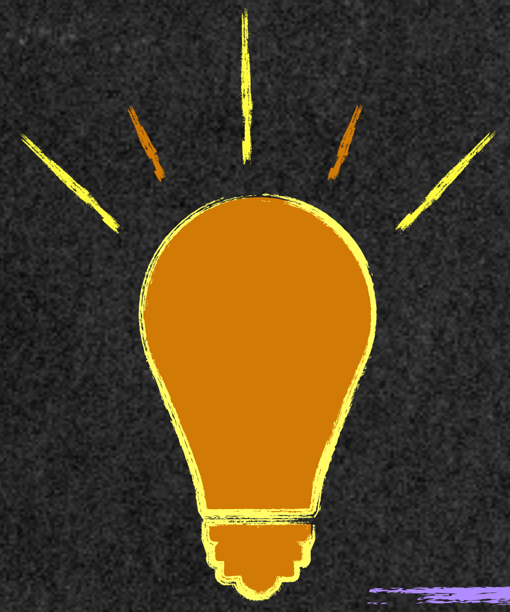
- Study asymptotic properties on these dynamical graphs

⇝ Phase transition for the weak giant component — Roberts & Şengül, 2018

Work in progress

⇝ Characterization of the metastable density of the contact process — Jacob - Linker & Mörters, 2019 & 2022; Linker & Remenik, 2020; da Silva - Oliveira & Valesin, 2021

Work done



OF PROOF

- Convergence in law

# of Proof (Convergence in Law)

Theorem D., Jacob – 2022+

$$\mathrm{DER}(\kappa,\beta) \xrightarrow[n \longrightarrow +\infty]{\text{loc}} \mathrm{GSPGW}(\kappa,\beta)$$

Coupling Method

times $\underset{n\to+\infty}{\longrightarrow} +\infty$

distances $\underset{n\to+\infty}{\longrightarrow} +\infty$

Theorem D., Jacob – 2022+

Assume that $\left(\kappa\left(1+\beta t_n\right)\right)^{d_n+2} = o(n)$ as $n \to +\infty$, then we can couple $\left(\mathcal{G}_t^{(n),\kappa,\beta}, o\right)_{t\geq 0}$ with $\mathrm{GSPGW}(\kappa,\beta)$ so that:

$$\mathbb{P}\left[\forall s < t_n : \left(\mathcal{B}_s^{d_n}\left(\mathcal{G}^{(n),\kappa,\beta}\right), o\right) = \mathcal{F}_s^{d_n}\right] = 1 - o(1)$$

# of Proof (Convergence in Law)

⇝ The dynamical ball of DER is a Markov process on $\mathbb{ST}^{d_n}$ (at least) until a random time $\tau_n \longrightarrow +\infty$ a.s.

- the evolutions of the ball and the graph formed by the other vertices are independent (independence of the evolution of the edges)

- the graph formed by the vertices outside the ball is an ER (stationarity property)

# of Proof (Convergence in Law)

⤳ The dynamical ball of DER is a Markov process on $\mathbb{ST}^{d_n}$

⤳ The transition rates of the ball of DER are close to those of the ball of GSPGW(κ,β)

- "splitting operation" : compare $\beta\left(1-\frac{\kappa}{n}\right)$ & β
- "growing operation" : compare ER & PGW
- "bad operation" : compare $\frac{\kappa}{n}$ & 0

# of Proof (Convergence in Law)

⟿ The dynamical ball of DER is a Markov process on $\mathbb{ST}^{d_n}$

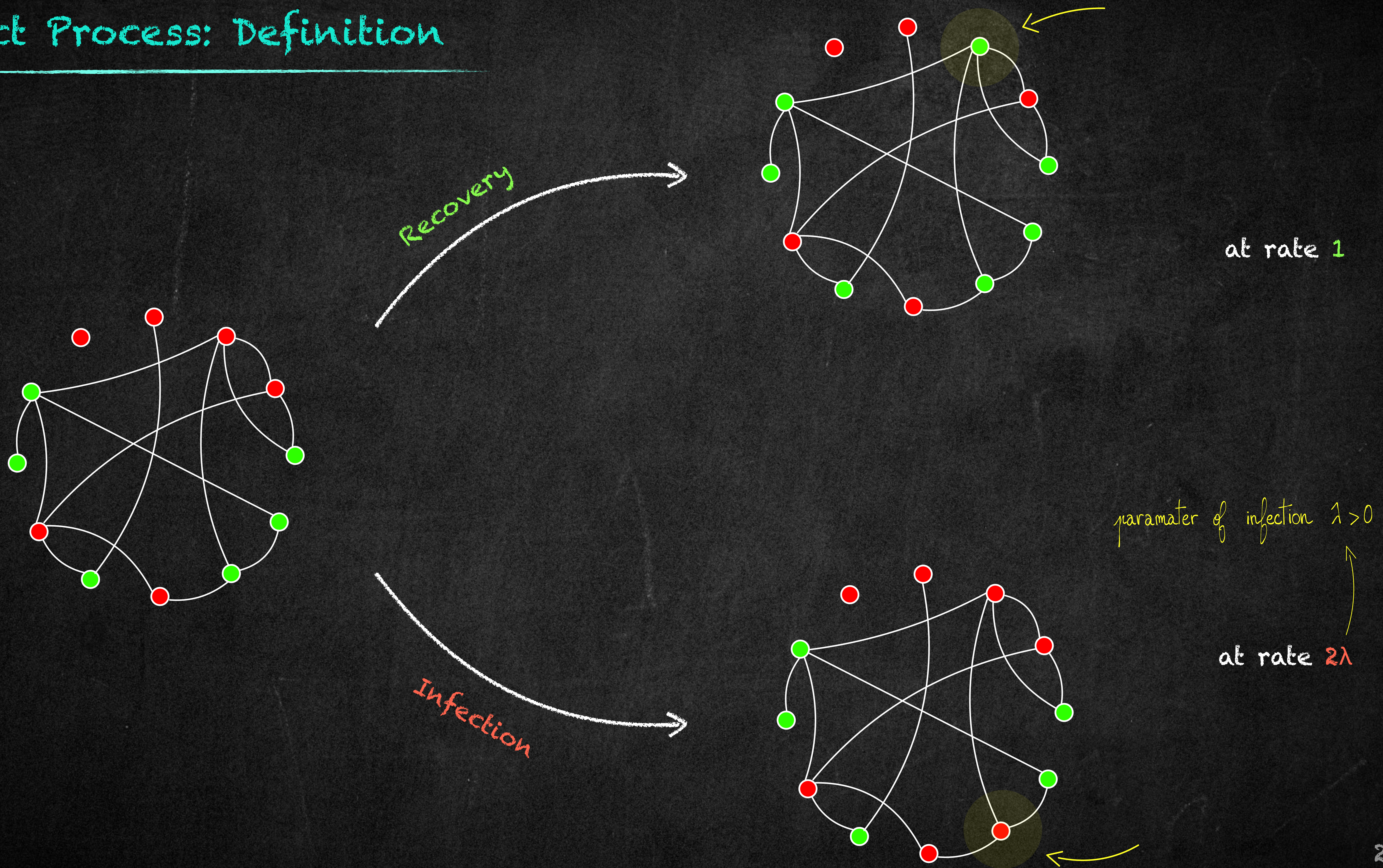
⟿ The transition rates of the ball of DER are closed to those of the ball of GSPGW(κ,β)

⟿ **Lemma:** If two Markov processes on a countable state space have "close" transition rates, then we can couple them s.t. they coincide until a random time which dominates an exponential variable.

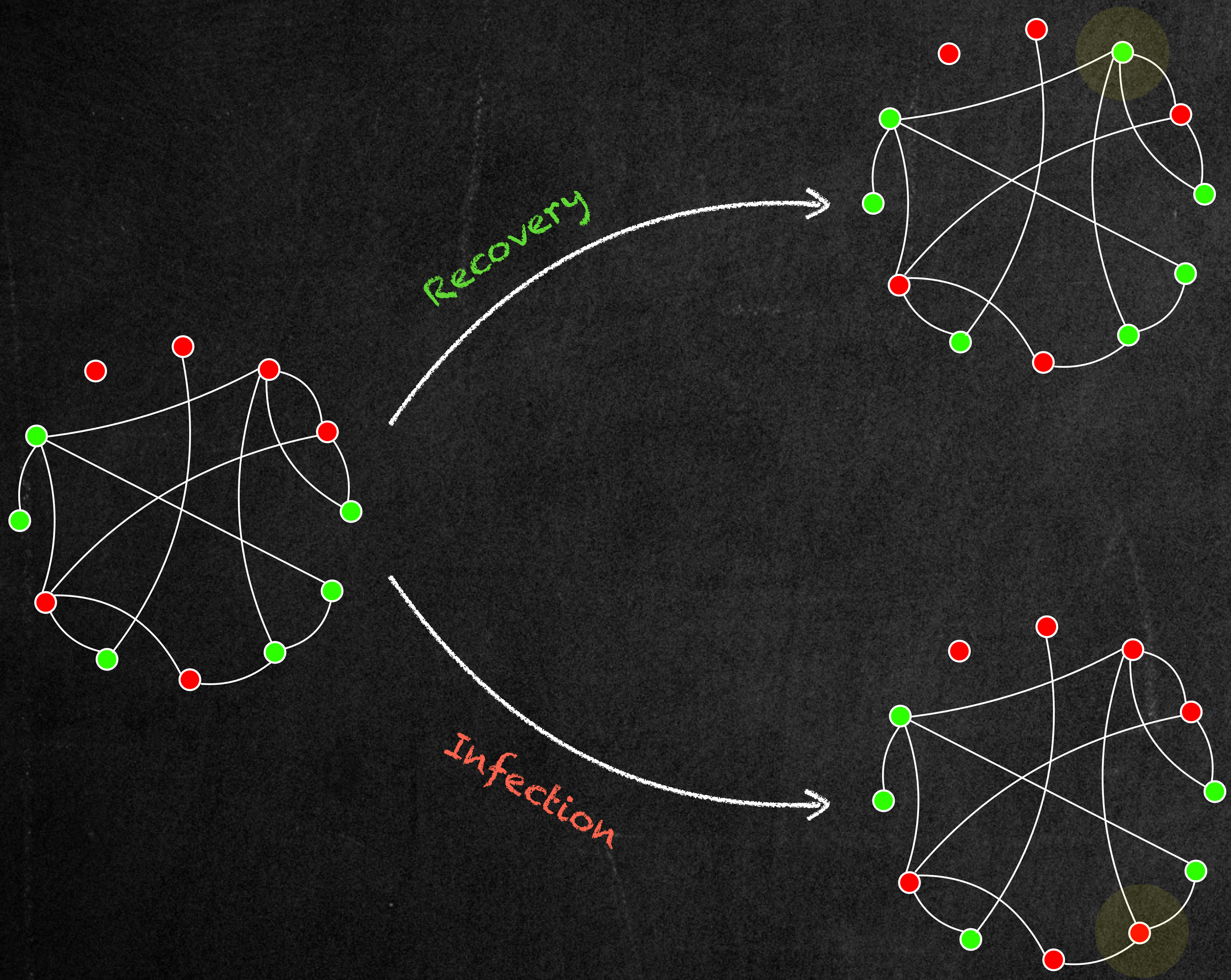
APPLICATION TO THE CONTACT PROCESS

- Definition
- Metastable Density

Contact Process: Definition



Contact Process: Interesting Questions



Let G_n be a graph with n vertices.

Begin with all vertices infected and set

$$\tau_{ext} := \inf \{ t \geq 0 : \xi_t^{V_n} = \emptyset \} .$$

Then

$$\mathbb{P} \left(\tau_{ext} < +\infty \right) = 1 .$$

Questions:

(1) Speed of extension ?

(2) Behaviour of $|\xi_{t_n}^{V_n}|$?

set of infected vertices at time t_n

Metastable Density of the Contact Process

Define

$$I_n(t) := \frac{1}{n}\,\mathbb{E}\left[\,|\xi_t^{V_n}|\,\right]$$ to be the expected density of infected vertices at time t.

Definition: We say that the contact process has a **METASTABLE DENSITY** $\rho(\lambda)$ if whenever t_n is going to infinity slower than exponentially, we have

$$\liminf_{n\to+\infty} I_n(t_n) = \limsup_{n\to+\infty} I_n(t_n) = \rho(\lambda).$$

Metastable Density of the Contact Process

Define

$$I_n(t) := \frac{1}{n}\,\mathbb{E}\left[\,|\xi_t^{V_n}|\,\right]$$ to be the expected density of infected vertices at time t.

Definition: We say that the contact process has a **METASTABLE DENSITY** $\rho(\lambda)$ if whenever t_n is going to infinity slower than exponentially, we have

$$\liminf_{n \to +\infty} I_n(t_n) = \limsup_{n \to +\infty} I_n(t_n) = \rho(\lambda).$$

Proposition: D. , Jacob - 2022+

Consider a contact process on a DIRG(κ,β) with some assumptions on the kernels κ and β.
Then $\rho(\lambda)$ exists.