Large deviations for the SSEP in weak contact with reservoirs

Based on j.w. with A. Bouley and C. Landim

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Configuration $\eta \in \Omega := \{0, 1\}^{\Lambda_N}$, $\eta_x = 1$ for an occupied site, $\eta_x = 0$ for an empty site. Initially, $\eta_x = 1$ w.p. $\rho_0(x/N)$.

Stirring dynamics: particles jump at rate 1 to empty neighbors

The SSEP’s empirical measure, on a diffusive timescale,

$$\pi_{tN^2}^N = \frac{1}{N} \sum_{x=1}^{N} \eta_x(tN^2) \delta_{x/N}$$

converges in a weak sense to $\rho(t, u) du$, where $\rho$ is called the SSEP’s hydrodynamic limit, and is the solution to the heat equation

$$\begin{cases} 
\partial_t \rho = \partial_{uu} \rho \\
\rho(0, \cdot) = \rho_0 
\end{cases}$$
SSEP, NO BOUNDARY INTERACTION

Simulation by Hugo Dorfsman (\(\alpha = 1/3, \beta = 2/3, N = 1000\))

Particles reflected at the boundaries: Neumann boundary conditions

\[ \partial_u \rho(t, 0) = \partial_u \rho(t, 1) = 0. \]
To maintain the SSEP out of equilibrium, we put it in contact with infinite reservoirs with density $\alpha$ and $\beta$.

Particles are created at rate $\alpha$ and removed at rate $1 - \alpha$ at the left boundary. Same with $\beta$ on the right.

The hydrodynamic limit $\rho$ is then supplemented by Dirichlet boundary conditions:

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\begin{cases}
\partial_t \rho = \partial_{uu} \rho \\
\rho(0, \cdot) = \rho_0 \\
\rho(t, 0) = \alpha, \quad \rho(t, 1) = \beta.
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**SSEP, STRONG RESERVOIRS**

Simulation by Hugo Dorfsman \((\alpha = 1/3, \beta = 2/3, N = 1000)\)

▷ **Dirichlet** boundary conditions

\[ \rho(t, 0) = \alpha, \quad \rho(t, 1) = \beta. \]
We define stationary solution to the hydrodynamic limit

$$\rho^*(u) = \alpha + (\beta - \alpha)u, \quad u \in [0, 1],$$

and define the **product measure fitting** $\rho^*$,

$$\mu^* = \otimes_{x=1}^{N} Ber(\rho(x/N))$$

$\alpha = \beta$: **Equilibrium case**, $\mu^*$ is **reversible** w.r.t. the dynamics.

$\alpha \neq \beta$: the stationary state is non-explicit, but **approximated** by $\mu^*$. 
Weak boundaries

Weak interactions with the boundaries: particles removed and created with rate of order $N^{-\theta}$.

The hydrodynamic limit’s b.c. now depend on $\theta$:

$\triangleright \theta < 1$: Dirichlet, $\rho(t, 0) = \alpha$, $\rho(t, 1) = \beta$.

$\triangleright \theta = 1$: Robin, $\partial_u \rho(t, 0) = \rho(t, 0) - \alpha$, $\partial_u \rho(t, 1) = \beta - \rho(t, 1)$.

$\triangleright \theta > 1$: Neumann, $\partial_u \rho(t, 0) = \partial_u \rho(t, 1) = 0$.

For $\theta = 1$, the stationary profile becomes

$$\rho^*(u) = \alpha + \frac{\beta - \alpha}{3}(u + 1), \quad u \in [0, 1],$$
Simulation by Hugo Dorfsman ($\alpha = 1/3$, $\beta = 2/3$, $N = 1000$)

Robin boundary conditions

$$\partial_u \rho(t, 0) = \rho(t, 0) - \alpha, \quad \partial_u \rho(t, 1) = \beta - \rho(t, 1).$$
Dynamical large deviations, $\theta < 1$

Question: what is the probability to observe, for $N$ finite but very large, a macroscopic profile $\pi^N$ different from the hydrodynamic limit $\rho$?

For $\theta < 1$ (Dirichlet b.c.),

**Theorem (Bertini, De Sole, Gabrielli, Jona–Lasinio, Landim ’03)**

Fix an initial profile $\rho_0$. There exists a convex functional $I_T(\pi \mid \rho_0)$ such that for any closed set $C$ (resp. open set $O$) in the set of trajectories,

$$\limsup_{N \to \infty} \frac{1}{N} \mathbb{P}(\pi^N \in C) \leq - \inf_{\pi \in C} I_T(\pi \mid \rho_0)$$

and

$$\liminf_{N \to \infty} \frac{1}{N} \mathbb{P}(\pi^N \in O) \leq - \inf_{\pi \in O} I_T(\pi \mid \rho_0)$$
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Question: How is the **large deviations functional** characterized?

Define for any test function $H$

$$J_H(\pi) = \langle \pi_T, H_T \rangle - \langle \rho_0, H_0 \rangle - \int_0^T dt \langle \pi_t, \partial_t H_t + \Delta H_t \rangle$$

$$+ \beta \int_0^T dt \partial_u H_t(1) - \alpha \int_0^T dt \partial_u H_t(0) - \int_0^T dt \langle \rho_t (1 - \rho_t), (\partial_u H_t)^2 \rangle$$

If $\pi = \rho$ is the **solution to the hydrodynamic limit** with Dirichlet b.c., then

$$J_H(\rho) = - \int_0^T dt \langle \rho_t (1 - \rho_t), (\partial_u H_t)^2 \rangle$$

One then defines the rate function

$$I_T(\pi) = \sup_H J_H(\pi).$$
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For \( \theta = 1 \) (Robin b.c.),

**Theorem (Franco, Gonçalves, Landim, Neumann ’22)**

Fix an initial profile \( \rho_0 \). There exists a convex functional \( I_T(\pi \mid \rho_0) \) such that for any closed set \( C \) (resp open set \( O \)) in the set of trajectories,

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\limsup_{N \to \infty} \frac{1}{N} \mathbb{P}(\pi^N \in C) \leq - \inf_{\pi \in C} I_T(\pi \mid \rho_0)
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and

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\liminf_{N \to \infty} \frac{1}{N} \mathbb{P}(\pi^N \in O) \leq - \inf_{\pi \in O} I_T(\pi \mid \rho_0)
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J_H(\pi) = \langle \pi_T, H_T \rangle - \langle \rho_0, H_0 \rangle - \int_0^T dt \langle \pi_t, \partial_t H_t + \Delta H_t \rangle \\
\int_0^T dt \pi_t(1) \partial_u H_t(1) - \int_0^T dt \pi_t(0) \partial_u H_t(0) \\
+ \int_0^T \Phi_{t, \alpha, \beta}(\pi_t(1), \pi_t(0), H_t(1), H_t(0)) - \int_0^T dt \langle \rho_t(1 - \rho_t), (\partial_u H_t)^2 \rangle
\]

For weak reservoir interactions, at the level of large deviations, the **boundary behavior is no longer fixed** and can fluctuate.

In both weak and strong cases, the minimizer \( H_\pi \) of \( J_H \) is a **weak driving force** that makes the deviation typical.
LARGE DEVIATION FUNCTIONAL

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J_H(\pi) = \langle \pi_T, H_T \rangle - \langle \rho_0, H_0 \rangle - \int_0^T dt \langle \pi_t, \partial_t H_t + \Delta H_t \rangle
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\int_0^T dt \pi_t(1) \partial_u H_t(1) - \int_0^T dt \pi_t(0) \partial_u H_t(0)
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For weak reservoir interactions, at the level of large deviations, the boundary behavior is no longer fixed and can fluctuate.

In both weak and strong cases, the minimizer \( H_\pi \) of \( J_H \) is a weak driving force that makes the deviation typical.
Question: what is the probability to observe in the stationary state $\mu^N$, for $N$ finite but very large, a profile $\gamma^N$ different from the stationary profile $\rho^*$?

For $\theta < 1$ (Dirichlet b.c.),

**Theorem (Bertini, De Sole, Gabrielli, Jona–Lasinio, Landim ’03)**

There exists a convex functional $S(\gamma)$ such that for any closed set $C$ (resp open set $O$) in the set of profiles, under the stationary state,

$$\limsup_{N \to \infty} \frac{1}{N} \mu^N (\gamma^N \in C) \leq - \inf_{\gamma \in C} S(\gamma)$$

and

$$\liminf_{N \to \infty} \frac{1}{N} \mu^N (\gamma^N \in O) \leq - \inf_{\gamma \in O} S(\gamma)$$

where the quasi potential is defined as

$$S(\gamma) = \lim_{T \to \infty} \sup_{\pi_T = \gamma, \pi_0 = \rho^*} I_T(\pi).$$
The quasi potential $S(\gamma)$

The quasi potential can be defined differently:

1) As the **entropy w.r.t. a product Bernoulli measure** fitting $\rho^*$

$$S(\gamma) = \int_0^1 du \left[ \gamma \log \left( \frac{\gamma}{\rho^*} \right) + (1 - \gamma) \log \left( \frac{1 - \gamma}{1 - \rho^*} \right) \right] (u).$$

2) As a variational problem $S(\gamma) = \inf_{f} \mathcal{G}(\gamma, f)$ whose minimizer $F = f_\gamma$ is solution to [Derrida Lebowitz Speer '02]

$$F'' = (\gamma - F) \frac{(F')^2}{F(1 - F)}$$

with b.c. $F(0) = \alpha, F(1) = \beta$. 
Question: what is the probability to observe in the stationary state $\mu^N$, for $N$ finite but very large, a profile $\gamma^N$ different from the stationary profile $\rho^*$?

For $\theta = 1$ (Robin b.c.),

**Theorem (Bouley, E’, Landim, ’22)**

There exists a convex functional $S(\gamma)$ such that for any closed set $C$ (resp open set $O$) in the set of profiles, under the stationary state,

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\limsup_{N \to \infty} \frac{1}{N} \mu^N (\gamma^N \in C) \leq - \inf_{\gamma \in C} S(\gamma)
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where the quasi potential is defined as

$$
S(\gamma) = \lim_{T \to \infty} \sup_{\pi_T = \gamma, \pi_0 = \rho^*} I_T(\pi).
$$
The quasi potential $S(\gamma)$

As in the case $\theta < 1$, one hopes to derive different formulations for the quasi potential.

1) The formulation as an entropy no longer holds.

2) However, still equal to a variational problem $S(\gamma) = \inf_f \mathcal{G}(\gamma, f)$ whose minimizer $F = f_\gamma$ is solution to [Derrida Lebowitz Speer ’02]

$$F'' = (\gamma - F) \frac{(F')^2}{F(1 - F)}$$

this time with **Robin** b.c. $F'(0) = F(0) - \alpha, F'(1) = \beta - F(1)$. 
THANKS FOR YOUR ATTENTION!

A few references:


▷ **Bouley, E’, Landim** (2022), *Steady state large deviations for one-dimensional, symmetric exclusion processes in weak contact with reservoirs*, preprint.

▷ **Franco, Gonçalves, Landim, Neumann** (2022), *Dynamical large deviations for the boundary driven symmetric exclusion process with Robin boundary conditions*, preprint.