Large deviations for the SSEP in weak contact with reservoirs

Based on j.w. with A. Bouley and C. Landim

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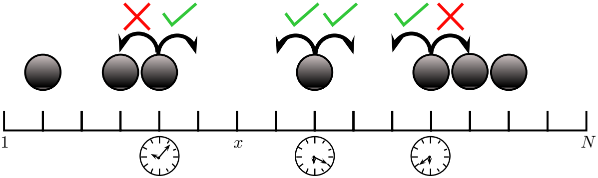
Symmetric Simple Exclusion Process (SSEP)



- ▷ Configuration $\eta \in \Omega := \{0,1\}^{\Lambda_N}$, $\eta_x = 1$ for an occupied site, $\eta_x = 0$ for an empty site. Initially, $\eta_x = 1$ w.p. $\rho_0(x/N)$.
- ▷ **Stirring dynamics**: particles jump at rate 1 to empty neighbors

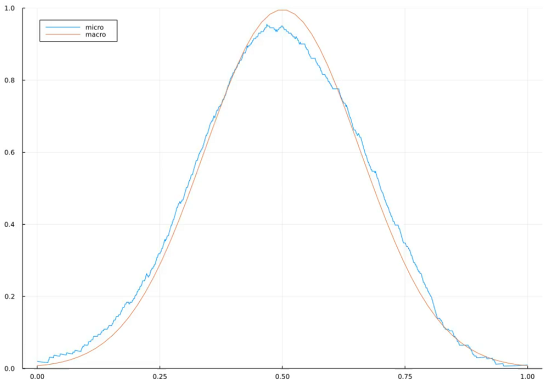
The SSEP's **empirical measure**, on a diffusive timescale,

$$\pi^N_{tN^2} = \frac{1}{N}\sum_{x=1}^{N} \eta_x(tN^2)\delta_{\frac{x}{N}}$$

converges in a weak sense to $\rho(t,u)du$, where ρ is called the SSEP's **hydrodynamic limit**, and is the solution to the heat equation

$$\begin{cases} \partial_t \rho = \partial_{uu}\rho \\ \rho(0,\cdot) = \rho_0 \end{cases}.$$

SSEP, NO BOUNDARY INTERACTION

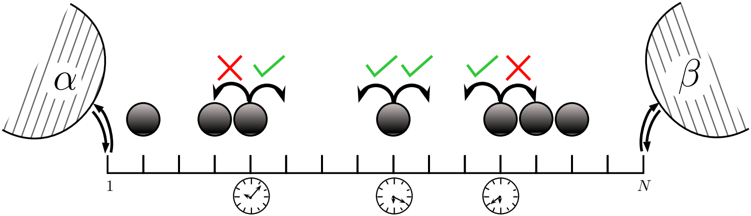


Simulation by Hugo Dorfsman ($\alpha = 1/3$, $\beta = 2/3$, $N = 1000$)

▷ Particles reflected at the boundaries: **Neumann** boundary conditions

$$\partial_u \rho(t, 0) = \partial_u \rho(t, 1) = 0.$$

Non-equilibrium SSEP



- To maintain the SSEP out of equilibrium, we put it in contact with infinite reservoirs with density α and β.
- particles are **created at rate** α and **removed at rate** $1-\alpha$ at the left boundary. Same with β on the right.

The hydrodynamic limit ρ is then supplemented by **Dirichlet boundary conditions**

$$\begin{cases} \partial_t \rho = \partial_{uu}\rho \\ \rho(0,\cdot) = \rho_0 \\ \rho(t,0) = \alpha, \quad \rho(t,1) = \beta. \end{cases}$$

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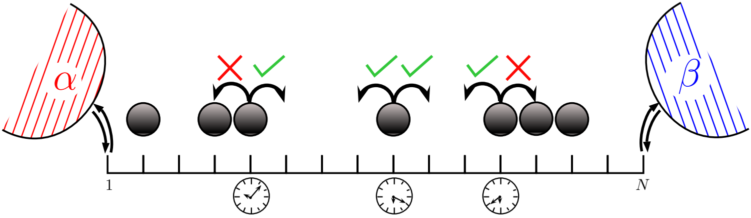

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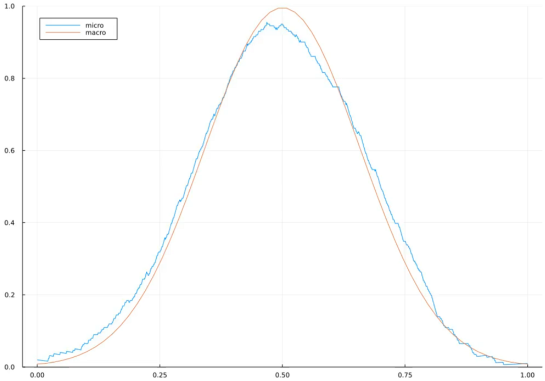
SSEP, STRONG RESERVOIRS



Simulation by Hugo Dorfsman ($\alpha = 1/3$, $\beta = 2/3$, $N = 1000$)

▷ **Dirichlet** boundary conditions

$$\rho(t,0) = \alpha, \qquad \rho(t,1) = \beta.$$

Hydrodynamic limit for boundary-driven SSEP

We define stationary solution to the hydrodynamic limit

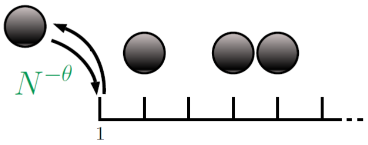
$$\rho^\star(u) = \alpha + (\beta - \alpha)u, \quad u \in [0,1],$$

and define the **product measure fitting** $\rho^\star$,

$$\mu^\star = \otimes_{x=1}^{N} Ber(\rho(x/N))$$

$\alpha = \beta$: **Equilibrium case** , $\mu^\star$ is **reversible** w.r.t. the dynamics.
$\alpha \neq \beta$: the stationary state is non-explicit, but **approximated** by $\mu^\star$.

WEAK BOUNDARIES



Weak interactions with the boundaries : particles removed and created with rate of order $N^{-\theta}$.

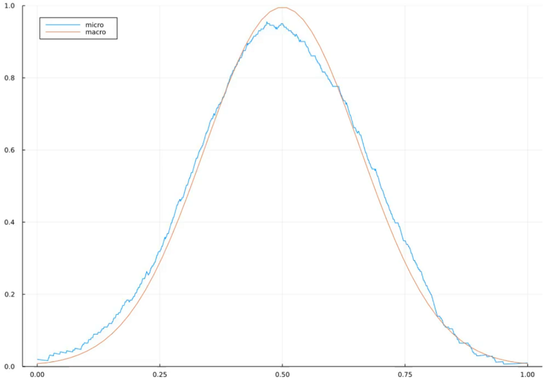
The hydrodynamic limit's b.c. now depend on θ :

- ▷ $\underline{\theta < 1}$: Dirichlet, $\rho(t,0) = \alpha, \quad \rho(t,1) = \beta$.
- ▷ $\underline{\theta = 1}$: Robin, $\partial_u \rho(t,0) = \rho(t,0) - \alpha, \qquad \partial_u \rho(t,1) = \beta - \rho(t,1)$.
- ▷ $\underline{\theta > 1}$: Neumann, $\partial_u \rho(t,0) = \partial_u \rho(t,1) = 0$.

For $\theta = 1$, the stationary profile becomes

$$\rho^\star(u) = \alpha + \frac{\beta - \alpha}{3}(u+1), \quad u \in [0,1],$$

SSEP, Weak reservoirs phase



Simulation by Hugo Dorfsman ($\alpha = 1/3$, $\beta = 2/3$, $N = 1000$)

▷ **Robin** boundary conditions

$$\partial_u \rho(t,0) = \rho(t,0) - \alpha, \qquad \partial_u \rho(t,1) = \beta - \rho(t,1).$$

Dynamical large deviations, $\theta < 1$

Question : what is the probability to observe, for N finite but very large, a macroscopic profile π^N different from the hydrodynamic limit ρ?

For $\theta < 1$ (Dirichlet b.c.),

Theorem (Bertini, De Sole, Gabrielli, Jona–Lasinio, Landim '03)

Fix an initial profile ρ_0. There exists a convex functional $I_T(\pi \mid \rho_0)$ such that for any closed set C (resp open set O) in the set of trajectories,

$$\limsup_{N\to\infty} \frac{1}{N} \mathbb{P}(\pi^N \in C) \leq - \inf_{\pi \in C} I_T(\pi \mid \rho_0)$$

and

$$\liminf_{N\to\infty} \frac{1}{N} \mathbb{P}(\pi^N \in O) \leq - \inf_{\pi \in O} I_T(\pi \mid \rho_0)$$

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Large deviation functional

Question : How is the **large deviations functional** characterized ?

Define for any test function H

$$J_H(\pi) = \langle \pi_T, H_T \rangle - \langle \rho_0, H_0 \rangle - \int_0^T dt \langle \pi_t, \partial_t H_t + \Delta H_t \rangle$$
$$+ \beta \int_0^T dt \partial_u H_t(1) - \alpha \int_0^T dt \partial_u H_t(0) - \int_0^T dt \langle \rho_t (1-\rho_t), (\partial_u H_t)^2 \rangle$$

If $\pi = \rho$ is the **solution to the hydrodynamic limit** with Dirichlet b.c., then

$$J_H(\rho) = -\int_0^T dt \langle \rho_t(1-\rho_t), (\partial_u H_t)^2 \rangle$$

One then defines the rate function

$$I_T(\pi) = \sup_H J_H(\pi).$$

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DYNAMICAL LARGE DEVIATIONS, $\theta = 1$

Question : what is the probability to observe, for N finite but very large, a macroscopic profile π^N different from the hydrodynamic limit ρ?

For $\theta = 1$ (Robin b.c.),

Theorem (Franco, Gonçalves, Landim, Neumann '22)

Fix an initial profile ρ_0. There exists a convex functional $I_T(\pi \mid \rho_0)$ such that for any closed set C (resp open set O) in the set of trajectories,

$$\limsup_{N\to\infty} \frac{1}{N} \mathbb{P}(\pi^N \in C) \leq - \inf_{\pi \in C} I_T(\pi \mid \rho_0)$$

and

$$\liminf_{N\to\infty} \frac{1}{N} \mathbb{P}(\pi^N \in O) \leq - \inf_{\pi \in O} I_T(\pi \mid \rho_0)$$

Large deviation functional

Question : How is the **large deviations functional** characterized ?

This time,

$$J_H(\pi) = \langle \pi_T, H_T \rangle - \langle \rho_0, H_0 \rangle - \int_0^T dt \langle \pi_t, \partial_t H_t + \Delta H_t \rangle$$

$$\int_0^T dt \pi_t(1) \partial_u H_t(1) - \int_0^T dt \pi_t(0) \partial_u H_t(0)$$

$$+ \int_0^T \Phi_{t,\alpha,\beta}(\pi_t(1), \pi_t(0), H_t(1), H_t(0)) - \int_0^T dt \langle \rho_t(1-\rho_t), (\partial_u H_t)^2 \rangle$$

For weak reservoir interactions, at the level of large deviations, the **boundary behavior is no longer fixed** and can fluctuate.

In both weak and strong cases, the minimizer H_π of J_H is a **weak driving force** that makes the deviation typical

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For weak reservoir interactions, at the level of large deviations, the **boundary behavior is no longer fixed** and can fluctuate.

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Static large deviations, $\theta < 1$

Question : what is the probability to observe **in the stationary state** μ^N, for N finite but very large, a profile γ^N different from the stationary profile $\rho^\star$?

For $\theta < 1$ (Dirichlet b.c.),

Theorem (Bertini, De Sole, Gabrielli, Jona–Lasinio, Landim '03)

There exists a convex functional $S(\gamma)$ such that for any closed set C (resp open set O) in the set of profiles, under the stationary state,

$$\limsup_{N\to\infty} \frac{1}{N} \mu^N(\gamma^N \in C) \leq - \inf_{\gamma \in C} S(\gamma)$$

and

$$\liminf_{N\to\infty} \frac{1}{N} \mu^N(\gamma^N \in O) \leq - \inf_{\gamma \in O} S(\gamma)$$

where the quasi potential is defined as

$$S(\gamma) = \lim_{T\to\infty} \sup_{\pi_T=\gamma, \pi_0=\rho^\star} I_T(\pi).$$

The quasi potential $S(\gamma)$

The quasi potential can be defined differently :

1) As the **entropy w.r.t. a product Bernoulli measure** fitting $\rho^\star$

$$S(\gamma) = \int_0^1 du \left[\gamma \log\left(\frac{\gamma}{\rho^\star}\right) + (1-\gamma)\log\left(\frac{1-\gamma}{1-\rho^\star}\right)\right](u).$$

2) As a variational problem $S(\gamma) = \inf_f \mathcal{G}(\gamma, f)$ whose minimizer $F = f_\gamma$ is solution to **[Derrida Lebowitz Speer '02]**

$$F'' = (\gamma - F)\frac{(F')^2}{F(1-F)}$$

with b.c. $F(0) = \alpha$, $F(1) = \beta$.

Static large deviations, $\theta = 1$

Question : what is the probability to observe **in the stationary state** μ^N, for N finite but very large, a profile γ^N different from the stationary profile $\rho^\star$?

For $\theta = 1$ (Robin b.c.),

Theorem (Bouley, E', Landim, '22)

There exists a convex functional $S(\gamma)$ such that for any closed set C (resp open set O) in the set of profiles, under the stationary state,

$$\limsup_{N\to\infty} \frac{1}{N}\mu^N(\gamma^N \in C) \leq -\inf_{\gamma\in C} S(\gamma)$$

and

$$\liminf_{N\to\infty} \frac{1}{N}\mu^N(\gamma^N \in O) \leq -\inf_{\gamma\in O} S(\gamma)$$

where the quasi potential is defined as

$$S(\gamma) = \lim_{T\to\infty} \sup_{\pi_T=\gamma, \pi_0=\rho^\star} I_T(\pi).$$

The quasi potential $S(\gamma)$

As in the case $\theta < 1$, one hopes to derive different formulations for the quasi potential.

1) The formulation as an entropy no longer holds.

2) However, still equal to a variational problem $S(\gamma) = \inf_f \mathcal{G}(\gamma, f)$ whose minimizer $F = f_\gamma$ is solution to **[Derrida Lebowitz Speer '02]**

$$F'' = (\gamma - F)\frac{(F')^2}{F(1-F)}$$

this time with **Robin** b.c. $F'(0) = F(0) - \alpha$, $F'(1) = \beta - F(1)$.

Thanks for your attention !

A few references:

- **Derrida, Lebowitz, Speer** (2002), *Large Deviation of the Density Profile in the Steady State of the Open Symmetric Simple Exclusion Process*, in J. Stat Phys. Vol. 107, 599–634.

- **Bertini, De Sole, Gabrielli, Jona-Lasinio and Landim** (2003), *Large deviations for the boundary driven symmetric simple exclusion process,* in Mathematical Phys., Analysis and Geom. Vol. 6, Issue 3, 231–267.

- **Bouley, E', Landim** (2022), *Steady state large deviations for one-dimensional, symmetric exclusion processes in weak contact with reservoirs*, preprint.

- **Franco, Gonçalves, Landim, Neumann** (2022), *Dynamical large deviations for the boundary driven symmetric exclusion process with Robin boundary conditions*, preprint.