La marche aléatoire de l'éléphant

Journées MAS

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Présentation réalisée avec le soutien du projet ERC COMBINEPIC

1. The elephant random walk

2. An elephant inside an urn ?

3. An elephant in a tree ?

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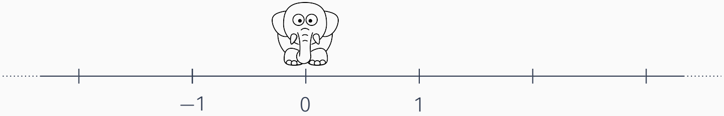

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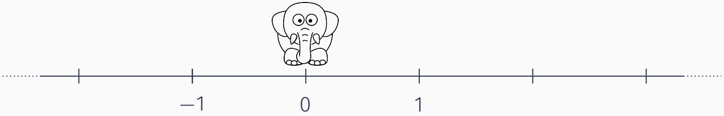
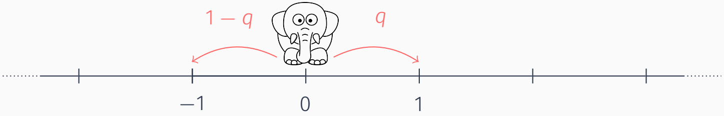


A time $n = 1$



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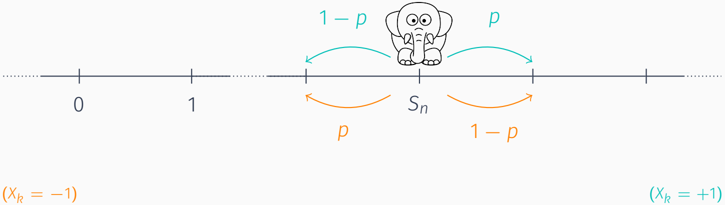

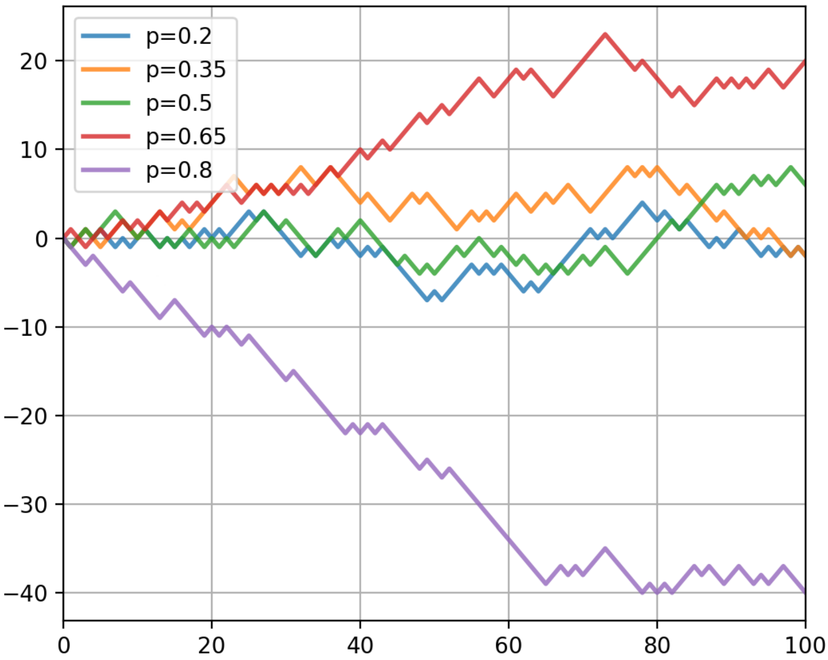

p=0.2
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2017 Colleti, Schütz et al. – Strong invariance principle and CLT
2018 Bercu – A martingale approach (LLN, LIL, QSL and CLT)
Businger - The Shark Random Swim
2019 Bercu, Chabanol, Ruch – Hypergeometric identities from the ERW
Bercu and L. – Multidimensional ERW
Kubota and Takei – Gaussian fluctuations in the superdiffusive regime
Vázquez Guevara – Almost sure CLT
2020 Bertenghi – Functional limit theorems for the MERW
Bertoin – Counterbalancing steps at random in a random walk
Baur – A class of reinforced RW (including ERW) using Pòlya-type urns
Bercu and L. – The center of mass of the ERW
2021 Bertoin – Counting the zeros of the ERW
L. – New insights on the RERW using a martingale approach
2022 5 more so far !

A martingale approach

We can write $X_{n+1} = \alpha_{n+1} X_{\beta_{n+1}}$ where the random variables

$$\alpha_{n+1} \sim \mathcal{R}(p), \qquad \beta_{n+1} \sim \mathcal{U}(1, \ldots, n)$$

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Then,

$$\begin{aligned}\mathbb{P}(X_{n+1} = 1 | \mathcal{F}_n) &= p \frac{\#\{\text{steps to the right}\}}{n} + (1-p) \frac{\#\{\text{steps to the left}\}}{n} \\ &= \frac{1}{2}\left(1 + (2p-1)\frac{S_n}{n}\right).\end{aligned}$$

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The conditional distribution of X_{n+1} given the past is

$$\mathcal{L}(X_{n+1} | \mathcal{F}_n) = \mathcal{R}(p_n)$$

where $p_n = \frac{1}{2}\Big(1 + a\frac{S_n}{n}\Big)$ and $a = 2p - 1$.

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We deduce that

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The study relies on a martingale approach

$$M_n = a_n S_n$$

where $a_1 = 1$ and

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The process (M_n) is a locally bounded square-integrable martingale. Indeed,

$$\mathbb{E}[M_{n+1}|\mathcal{F}_n] = a_{n+1}\mathbb{E}[S_{n+1}|\mathcal{F}_n] = a_{n+1}\gamma_n S_n = a_n S_n = M_n$$

and $\mathbb{E}[M_n^2] \leq (na_n)^2$.

Three regimes

We find that

$$\langle M \rangle_n = \sum_{k=1}^{n} a_k^2 - a^2 \sum_{k=1}^{n} a_k^2 \left(\frac{S_k}{k} \right)^2.$$

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- the critical regime where $a = 1/2$ and $v_n = O(\log n)$,
- the superdiffusive regime where $a > 1/2$ and $v_n = O(1)$.

Main results

Theorem (Law of large numbers)

Diffusive	*Critical*	*Superdiffusive*
$\lim_{n\to\infty} \frac{S_n}{n} \overset{a.s.}{=} 0$	$\lim_{n\to\infty} \frac{S_n}{\sqrt{n}\log n} \overset{a.s.}{=} 0$	$\lim_{n\to\infty} \frac{S_n}{n^a} \overset{a.s./\mathbb{L}^4}{=} L$

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Theorem (Quadratic strong law and law of iterated logarithm)

Diffusive

$$\lim_{n\to\infty} \frac{1}{\log n} \sum_{k=1}^{n} \left(\frac{S_k}{k}\right)^2 \overset{a.s.}{=} \frac{1}{1-2a}$$

$$\limsup_{n\to\infty} \frac{S_n^2}{2n \log\log n} \overset{a.s.}{=} \frac{1}{1-2a}$$

Critical

$$\lim_{n\to\infty} \frac{1}{\log\log n} \sum_{k=1}^{n} \left(\frac{S_k}{k \log k}\right)^2 \overset{a.s.}{=} 1$$

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Theorem (Asymptotic normality)

Diffusive

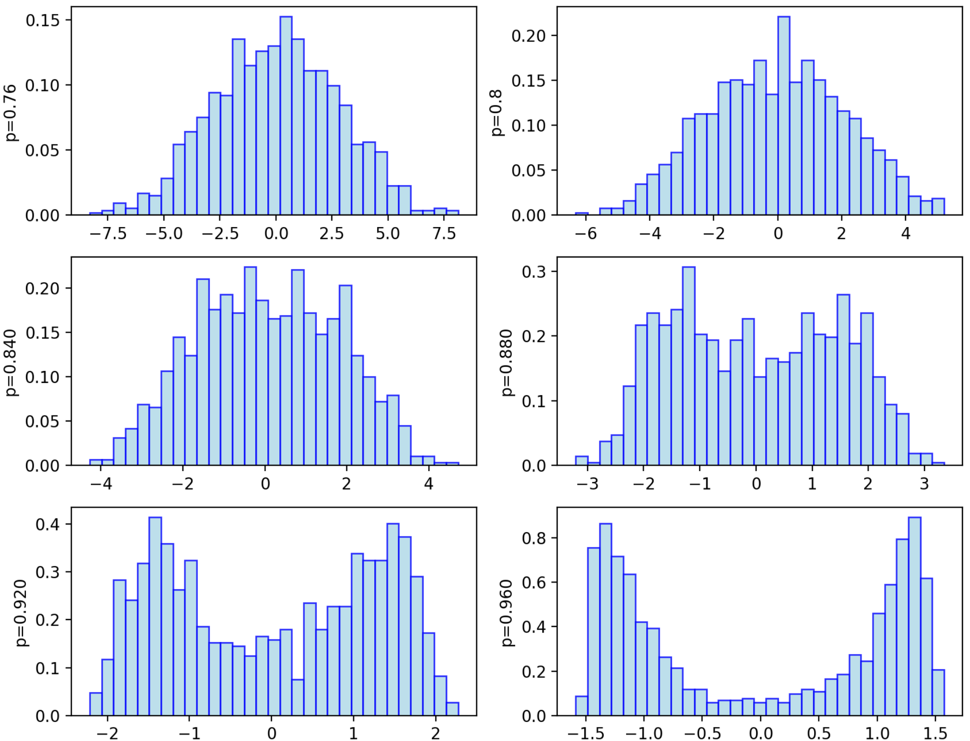
$$\frac{S_n}{\sqrt{n}} \xrightarrow[n\to\infty]{\mathcal{L}} \mathcal{N}\left(0, \frac{1}{1-2a}\right)$$

Critical

$$\frac{S_n}{\sqrt{n\log n}} \xrightarrow[n\to\infty]{\mathcal{L}} \mathcal{N}(0,1)$$

Superdiffusive

$$\frac{S_n - n^a L}{\sqrt{n}} \xrightarrow[n\to\infty]{\mathcal{L}} \mathcal{N}\left(0, \frac{1}{2a-1}\right)$$


p=0.76
p=0.8
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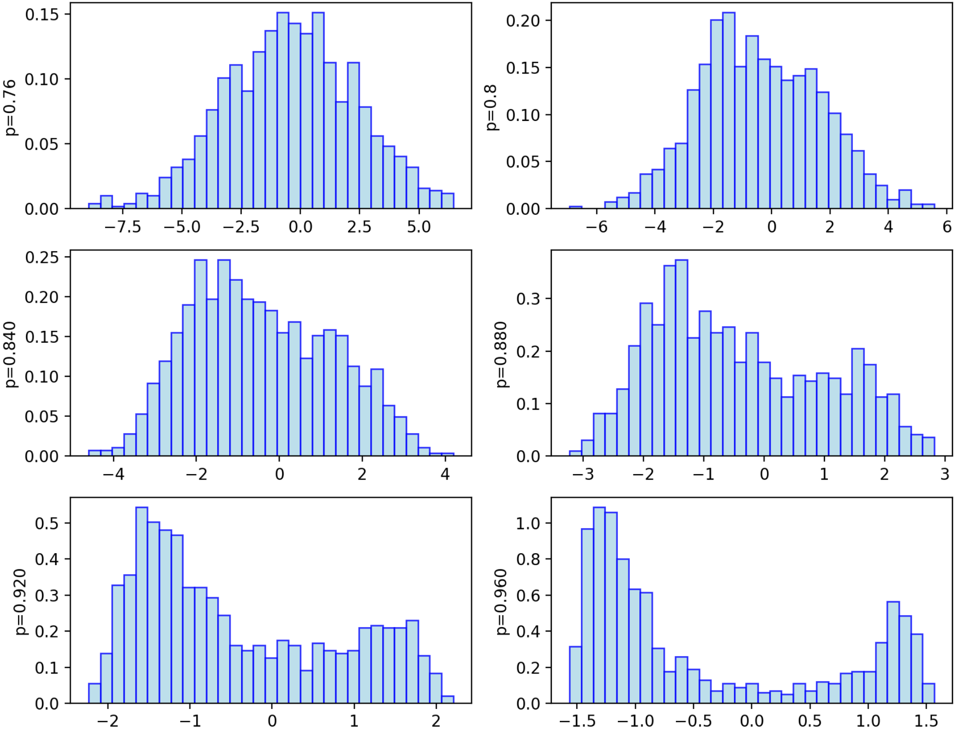
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An elephant inside an urn ?

Pólya urn processes

At the inital time $n = 0$, an urn is filled with $\alpha \geq 0$ red balls and $\beta \geq 0$ blue balls. Then, at any time $n \geq 1$ one ball is picked randomly from the urn and its color observed. If it is red (blue) it is then returned to the urn together with a additional red (c red) balls and $b \geq 0$ blue ($d \geq 0$ blue) ones.

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We assume that the urn is balanced, $\boldsymbol{S = a + b = c + d \geq 1}$. S is the maximum eigenvalue of R and the second eigenvalue of R is given by $\boldsymbol{m = a - c = d - b}$, with respective eigenvectors

$$v_1 = \frac{S}{b+c}\begin{pmatrix} c \\ b \end{pmatrix} \qquad \text{and} \qquad v_2 = \frac{S}{b+c}\begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

We denote $\boldsymbol{\sigma = m/S < 1}$ the ratio of the two eigenvalues.

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What about $0 < p < 1$?

A more generalized model of Pólya urns

S. Janson – Functional limit theorems for multitype branching processes and generalized Pólya urns. *Stochastic Processes and their Applications 110* (2004)

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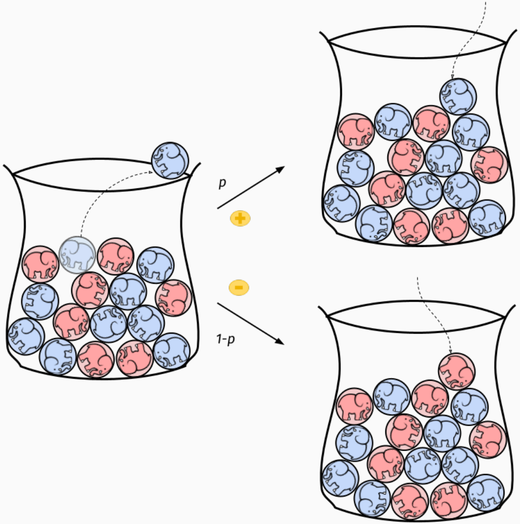
In this case the replacement matrix R becomes the mean replacement matrix A such that

$$A = \left(\mathbb{E}[\theta_{i,j}]\right)_{1 \leq i,j \leq q}$$

where q is the number of colors and $\theta_{i,j}$ is the random variable saying how many balls of type j are added when a ball of type i is picked.

The ERW and the associated Pólya urn

Let $U_n = \begin{pmatrix} R_n \\ B_n \end{pmatrix}$ be an urn filled with red and blue balls. We make the following connection :



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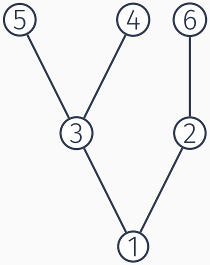
In this case, S_n has the same distribution as $R_n - B_n = 2R_n - n$.

An elephant in a tree ?

Random recursive tree and Bernoulli percolation



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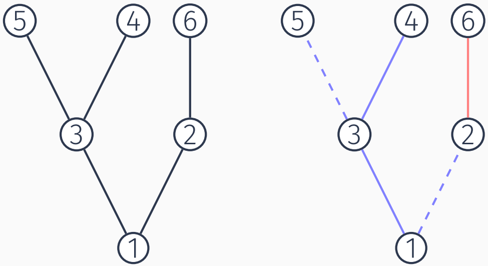

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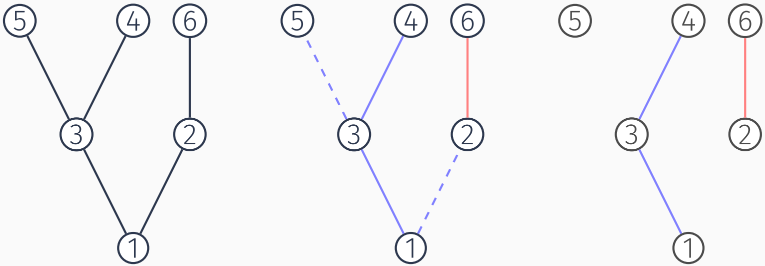

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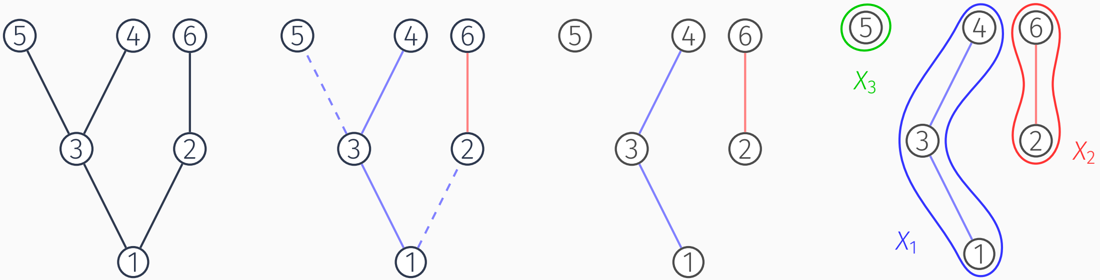

Other definition of the ERW

Let (X_n) be a sequence of i.i.d. random variables with law $\mathcal{R}(1/2)$ and (ε_n) a sequence of i.i.d. Bernoulli random variables with parameter a. Then, set $\hat{X}_1 = X_1$ and, for $n \geq 1$, choose an instant k among the previous instants such that

$$\hat{X}_{n+1} = \begin{cases} X_{1+\sigma(n+1)} & \text{if } \varepsilon_{n+1} = 0, \\ \hat{X}_k & \text{if } \varepsilon_{n+1} = 1, \end{cases}$$

where $\sigma(n) = \sum_{k=2}^{n}(1 - \varepsilon_k)$ is counting the number of inovations up to time $n \geq 2$.

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Kürsten explained that the sequence

$$\hat{S}_n = \hat{X}_1 + \ldots + \hat{X}_n,$$

is the elephant random walk with memory parameter $p \in (1/2, 1)$.

Other definition of the ERW

Let (X_n) be a sequence of i.i.d. random variables with law $\mathcal{R}(1/2)$ and (ε_n) a sequence of i.i.d. Bernoulli random variables with parameter a. Then, set $\hat{X}_1 = X_1$ and, for $n \geq 1$, choose an instant k among the previous instants such that

$$\hat{X}_{n+1} = \begin{cases} X_{1+\sigma(n+1)} & \text{if } \varepsilon_{n+1} = 0, \\ \hat{X}_k & \text{if } \varepsilon_{n+1} = 1, \end{cases}$$

where $\sigma(n) = \sum_{k=2}^{n}(1-\varepsilon_k)$ is counting the number of inovations up to time $n \geq 2$.

Kürsten explained that the sequence

$$\hat{S}_n = \hat{X}_1 + \ldots + \hat{X}_n,$$

is the elephant random walk with memory parameter $p \in (1/2, 1)$. Consequently, if $a > 1/2$ ($p > 3/4$) we know that

$$\lim_{n\to\infty} \frac{\hat{S}_n}{n^a} = L \quad \text{a.s. and in } \mathbb{L}^2.$$

ERW, random recursive trees and Bernoulli percolation

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in the way that

$$\hat{S}_n = \sum_{i=1}^{\infty} |c_n(i)| X_i$$

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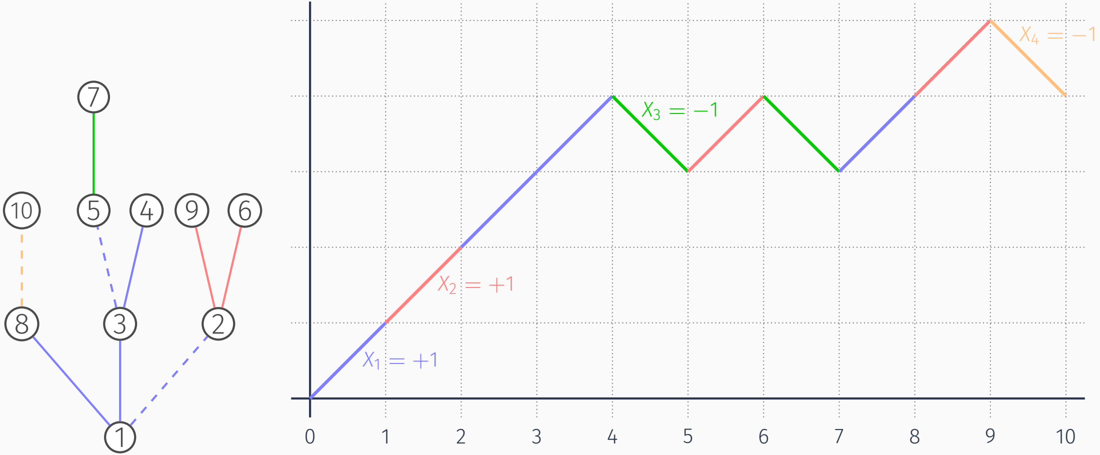
$$|c_{n+1}(i)| = \begin{cases} 0 & \text{if } n < \tau_i, \\ 1 & \text{if } n = \tau_i, \\ |c_n(i)| + \mathbb{1}_{\hat{X}_{n+1} = X_i} \mathbb{1}_{\varepsilon_{n+1} = 1} & \text{if } n > \tau_i. \end{cases}$$

An example



New insights on L

It has been proved that (see e.g. Baur and Bertoin, 2016)

$$\lim_{n\to\infty} \frac{|c_n(i)|}{n^a} = \Gamma_i \quad \text{a.s.}$$

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such that Γ_1 has a Mittag-Leffler distribution with parameter a and Γ_i a random variable with the same law as $(\beta_{\tau_i})^a \cdot \Gamma_1$, where β_i denotes a beta variable with parameter $(1, i-1)$ and is further independent of Γ_1.

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Consequently, it is possible to obtain the following decomposition of L

The distribution of L

$$L = \sum_{i=1}^{\infty} \Gamma_i \cdot X_i.$$

Merci pour votre attention !