Estimation dans les modèles de Hawkes avec inhibition

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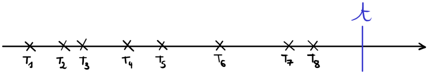
Point process

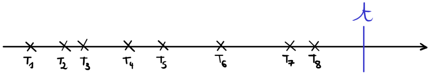


- $(T_k)_{k \geqslant 1}$ a random collection of points
- Intensity function $\lambda(t)$: immediate probability of observing an event at time t

Hawkes process

The intensity function depends on the past history $\mathcal{H}_t$.

Linear univariate Hawkes process



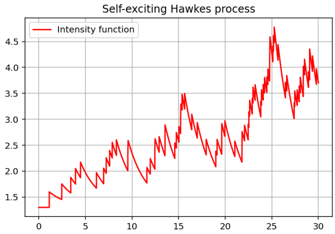
(Conditional) intensity function

$$\lambda(t|\mathcal{H}_t) = \lambda(t) = \mu + \sum_{T_k \leqslant t} h(t - T_k)$$

- μ is called the baseline
- h describes the dependence between points

Modeling self-exciting phenomena

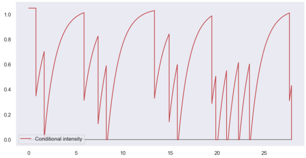
$$\lambda(t) = \mu + \sum_{T_k \leqslant t} h(t - T_k)$$



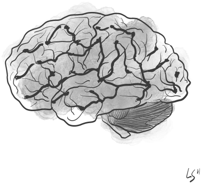
- $h \geqslant 0$ which ensures that λ is positive
- An event increases the probability of observing another event
- Applications: seismology, epidemiology

Motivation for modeling inhibition

- Recent interest for modeling inhibition: an event decreases the probability of observing another event



- Applications in neuroscience (multivariate setting)



How to model inhibition

- Using a link function

$$\lambda(t) = \phi\left(\mu + \sum_{T_k \leqslant t} h(t - T_k)\right)$$

for instance $\phi(x) = x_+ = max(x, 0)$.

- Other modelings:
  - Multiplicative inhibtion (Duval et al. 2022)
  - Self-limiting process (Olinde and Short, 2020)

Inference methods

- Bayesian procedures : Nonparametric (Sulem et al (2021)) and parametric (Deutsch (2022))
- Other inference procedures not specifically designed for handling non-linear intensities, among them:
  - ML approaches with exponential kernel (Lemonnier 2014)
  - Nonparametric least-squares estimator (Reynaud-Bouret et al, 2014)
  - Parametric least-square estimator (Bacry et al. 2018)

Approximated MLE

$$\lambda(t) = \left(\mu + \sum_{Tk \leqslant t} h(t - T_k)\right)_+$$

Log-likelihood (Daley and Vere-Jones, 2003)

$\mathcal{P} = \{\lambda_\theta, \theta \in \Theta\}$, for instance $h(t) = \alpha \exp(-\beta t)$ and $\theta = (\mu, \alpha, \beta)$

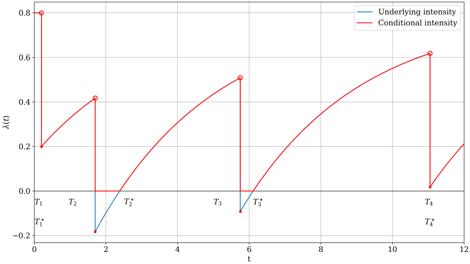
$$\ell_t(\theta) = \sum_{k=1}^{N(t)} \log\left(\lambda_\theta(T_k^-)\right) - \underbrace{\int_0^t \lambda_\theta(u)\,\mathrm{d}u}_{\text{compensator } \Lambda_\theta(t)}$$

Approximated version (Lemonnier et al. 2014)

- $\lambda(t) \simeq \mu + \sum_{T_k \leqslant t} h(t - T_k)$
- Write and optimize $\ell_t(\theta)$ allowing negative coefficients for α

Underlying intensity and restart times

$$\lambda(t) = \left(\mu + \sum_{Tk \leqslant t} h(t - T_k)\right)_+$$



- Underlying intensity $\lambda^\star$: $\lambda(t) = (\lambda^\star(t))_+$
- Restart times: $T_k^\star = \inf\{t > T_k | \lambda(t) > 0\}$

Exact MLE

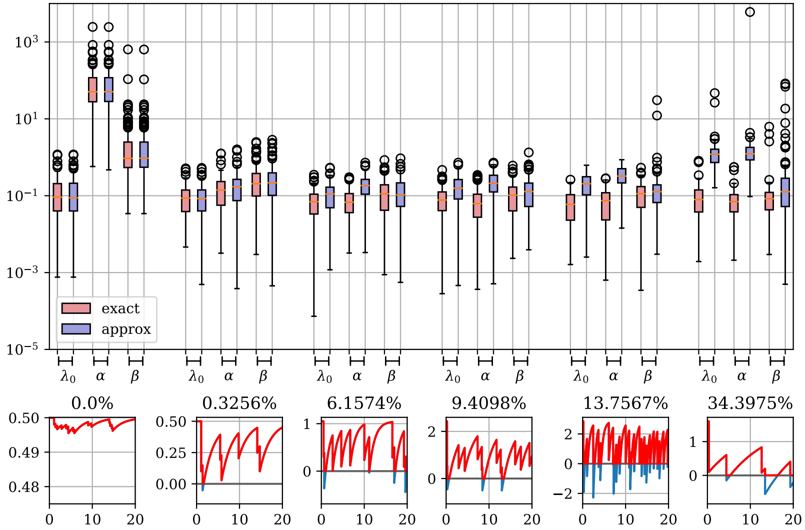
Reminder

$$\ell_t(\theta) = \sum_{k=1}^{N(t)} \log\left(\lambda_\theta(T_k^-)\right) - \underbrace{\int_0^t \lambda_\theta(u)\,\mathrm{d}u}_{\text{compensator } \Lambda_\theta(t)}$$

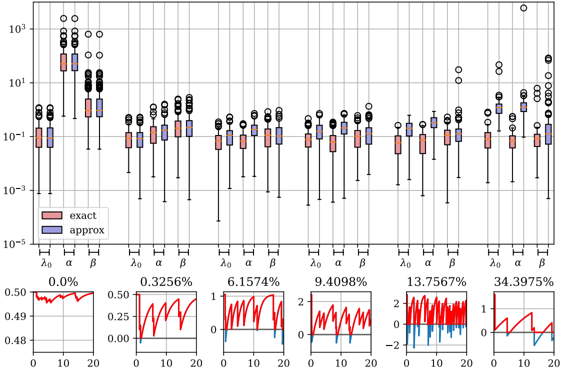
- **What do we do with the underlying intensity?**
  $\rightarrow$ Compute the intensity before jumps: $\lambda_\theta(T_k^-)$
- **What do we do with the restart times $T_k^\star$?**
  $\rightarrow$ Compute the compensator:

$$\Lambda(t) = \begin{cases} \lambda_0 t & \text{if } t < T_1 \\ \lambda_0 T_1 + \sum_{k=1}^{N(t)-1} \int_{T_k^\star}^{T_{k+1}} \lambda^\star(u)\,\mathrm{d}u + \int_{T_{N(t)}^\star}^{t} \lambda^\star(u)\,\mathrm{d}u & \text{if } t \geqslant T_1 \end{cases}$$

Comparison between exact and approximated MLE



Comparison between exact and approximated MLE

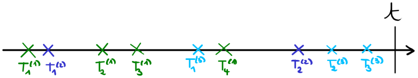


H_0: $(T_k)_{k\geqslant 1}$ is a realization of an exponential Hawkes process with parameters (μ, α, β)

p-values for a goodness-of-fit test (Daley and Vere-Jones, 2003)

Exact	0.78	0.72	0.69	0.73	0.73	0.70
Approx	0.78	0.70	0.55	0.51	0.29	5.12×10^{-6}

Multivariate Hawkes process



(Conditional) intensity functions

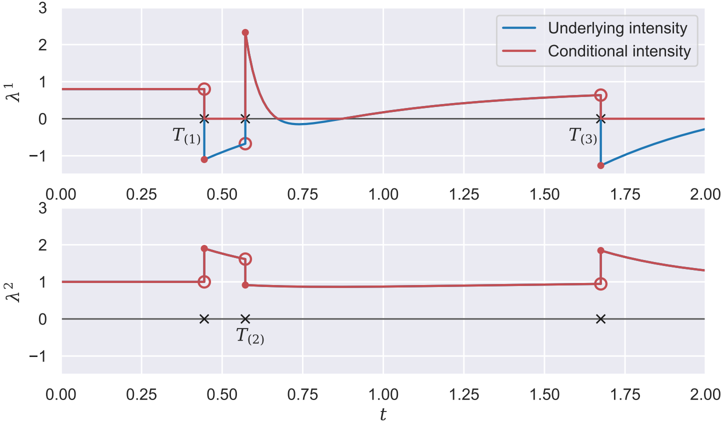
For $1 \leqslant i \leqslant q$,

$$\lambda^{(i)}(t) = \mu_i + \sum_{i=1}^{q} \sum_{T_k^j \leqslant t} h_{i,j}(t - T_k^j)$$

- μ_i baseline
- $h_{i,j}$ describes the dependence between points of types i and j

Example of a 2-dimensional Hawkes process

$$\alpha_{1,1}, \alpha_{2,2} < 0,\ \alpha_{1,2}, \alpha_{2,1} > 0$$



Proposition

If for each i, $\beta_{i,j} = \beta_i$ for all i, then $\lambda^{(i)}$ is monotone between two event times.

Numerical results in dimension 2

Compared methods

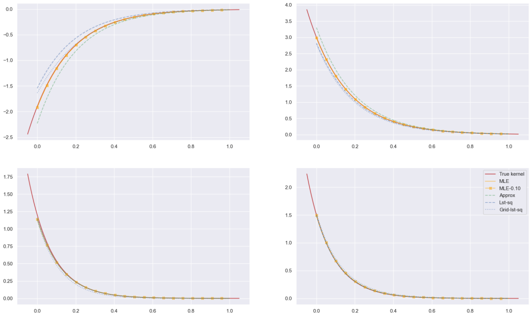
- MLE
- MLE-ϵ: cumulative thresholding
- Approximated MLE="approx"
- Approximated least-squares="Lst-sq"

Scenario 1:

$$\begin{pmatrix}\mu^1\\ \mu^2\end{pmatrix} = \begin{pmatrix}0.5\\ 1.0\end{pmatrix}$$

$$\begin{pmatrix}\alpha_{11} & \alpha_{12}\\ \alpha_{21} & \alpha_{22}\end{pmatrix} = \begin{pmatrix}-1.9 & 3.0\\ 1.2 & 1.5\end{pmatrix}$$

$$\begin{pmatrix}\beta_1\\ \beta_2\end{pmatrix} = \begin{pmatrix}5.0\\ 8.0\end{pmatrix}$$



Numerical results in dimension 2

Compared methods

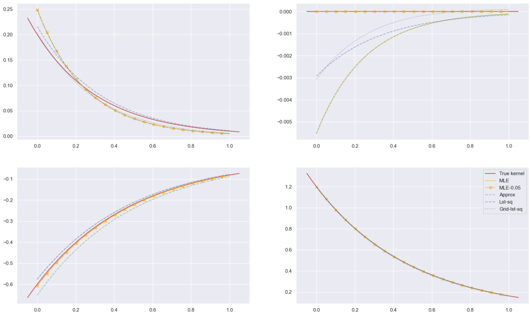
- MLE
- MLE-ϵ: cumulative thresholding
- Approximated MLE="approx"
- Approximated least-squares="Lst-sq"

Scenario 2:

$$\begin{pmatrix}\mu^1\\ \mu^2\end{pmatrix} = \begin{pmatrix}0.7\\ 1.0\end{pmatrix}$$

$$\begin{pmatrix}\alpha_{11} & \alpha_{12}\\ \alpha_{21} & \alpha_{22}\end{pmatrix} = \begin{pmatrix}0.2 & 0.0\\ -0.6 & 1.2\end{pmatrix}$$

$$\begin{pmatrix}\beta_1\\ \beta_2\end{pmatrix} = \begin{pmatrix}3.0\\ 2.0\end{pmatrix}$$



Numerical results in dimension 2

Compared methods

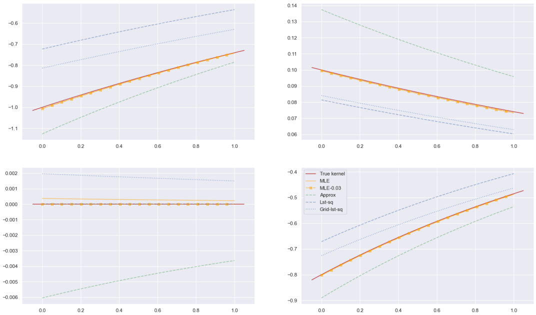
- MLE
- MLE-ϵ: cumulative thresholding
- Approximated MLE="approx"
- Approximated least-squares="Lst-sq"

Scenario 3:

$$\begin{pmatrix}\mu^1\\ \mu^2\end{pmatrix} = \begin{pmatrix}1.2\\ 1.0\end{pmatrix}$$

$$\begin{pmatrix}\alpha_{11} & \alpha_{12}\\ \alpha_{21} & \alpha_{22}\end{pmatrix} = \begin{pmatrix}-1.0 & 0.1\\ 0.0 & -0.8\end{pmatrix}$$

$$\begin{pmatrix}\beta_1\\ \beta_2\end{pmatrix} = \begin{pmatrix}0.3\\ 0.5\end{pmatrix}$$

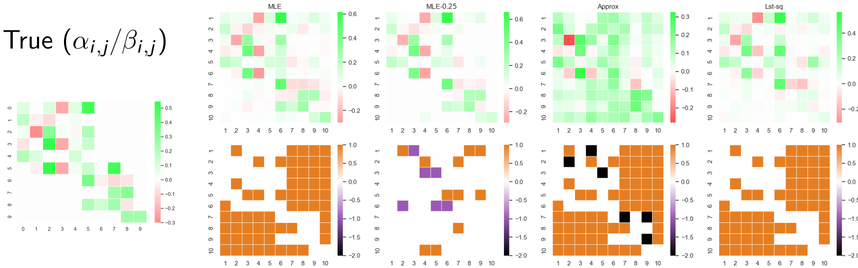


Summary of the numerical experiments

	Scenario (1)			Scenario (2)			Scenario (3)		
p-value	p_1	p_2	p_{tot}	p_1	p_2	p_{tot}	p_1	p_2	p_{tot}
True	0.49	0.44	0.43	0.54	0.47	0.48	0.51	0.62	0.34
MLE	0.44	0.44	0.40	0.48	0.46	0.48	0.55	0.64	0.36
MLE-ε	0.44	0.44	0.40	0.49	0.46	0.49	0.55	0.57	0.33
Approx	0.26	0.44	0.36	0.48	0.45	0.46	0.0	0.01	0.0
Lst-sq	0.15	0.44	0.39	0.53	0.46	0.48	0.0	0.0	0.0

- MLE and MLE-ε perform well in all scenarios
- MLE and MLE-ε outperform other methods when there are strong inhibiting effects
- MLE-ε allows to recover the non-null interactions

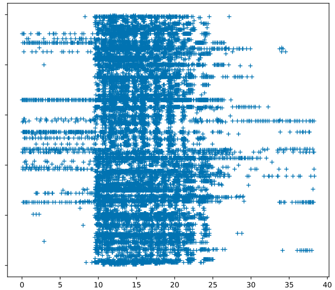
Numerical results for a 10-dimensional process



p-value	p_1	p_2	p_3	p_4	p_5	p_6	p_7	p_8	p_9	p_{10}	p_{tot}
True	0.40	0.71	0.27	0.73	0.55	0.39	0.70	0.23	0.67	0.63	0.46
MLE	0.41	0.74	0.29	0.64	0.46	0.38	0.64	0.24	0.62	0.12	0.336
MLE-ϵ	0.44	0.72	0.29	0.67	0.50	0.31	0.63	0.27	0.62	0.12	0.40
Approx	0.35	0.41	0.45	0.04	0.40	0.22	0.39	0.34	0.08	0.14	0.28
Lst-sq	0.43	0.71	0.20	0.60	0.45	0.31	0.61	0.04	0.57	0.04	0.28

Application to red-eared turtle neuronal data

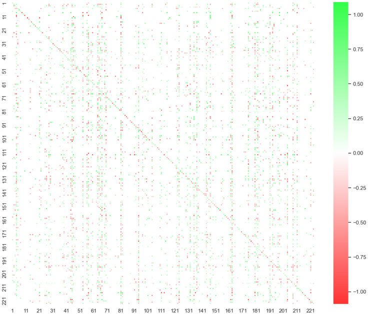
- Observations: Spike trains of 223 neurons of a red-eared turtle

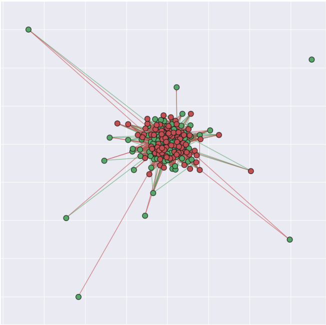


Question

Can we recover the interactions between neurons?

Heatmap after thresholding and connectivity graph





- We estimate (self and mutually) exciting and inhibiting effects
- We recover the connectivity graph were some neurons interact a lot and other are isolated

Future work

- Theoretical guarantees of the MLE estimator (extension to the existing work for self-exciting processes)
- Methodology: test procedures to detect the non-null interactions/post-hoc selection
- Neuronal applications: further investigation for interpretability

References

[1] Bonnet, A. and Martinez Herrera, M. and Sangnier, M. (2021). *Maximum likelihood estimation for Hawkes processes with self-excitation or inhibition*. Statistics & Probability Letters, Volume 179, 109214.

[2] Bonnet, A. and Martinez Herrera, M. and Sangnier, M. (2022). *Inference of multivariate exponential Hawkes processes with inhibition and application to neuronal activity*. arXiv:2205.04107