An alternative pricing method for financial models with transaction costs

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Joint work with Emmanuel Lépinette
We consider a portfolio process: \((V_t)_{t=-1}^T\) where \(V_{-1} \in \mathbb{R}e_1\) is the initial endowment expressed in cash.

The solvency set \(G_t : \Omega \rightarrow \mathbb{R}^d\) is \(\mathcal{F}_t\)-measurable random closed set, i.e.

\[
\text{Graph}(G_t) = \{(\omega, x) : x \in G_t(\omega)\} \in \mathcal{F}_t \otimes \mathcal{B}(\mathbb{R}^d)
\]

The cost process \(C = (C_t)_{t=0}^T\) associated to \(G\) is defined as:

\[
C_t(z) = \inf\{\alpha \in \mathbb{R} : \alpha e_1 - z \in G_t\} = \min\{\alpha \in \mathbb{R} : \alpha e_1 - z \in G_t\}
\]

\(C_t\) is lower semicontinuous and is allowed to be non-convex.

Similarly, we may define the liquidation value process \(L = (L_t)_{t=0}^T\):

\[
L_t(z) := \sup\{\alpha \in \mathbb{R} : z - \alpha e_1 \in G_t\}, \quad z \in \mathbb{R}^d.
\]
A portfolio process is a stochastic process \((V_t)_{t=-1}^T\) satisfying:

\[\Delta V_t = V_t - V_{t-1} \in -G_t, \text{ a.s., } t = 0, \cdots, T.\]

or equivalently, \(L_t(-\Delta V_t) \geq 0\) a.s.

Some examples:

\[C_t(z) = zS_t, \text{ no transaction cost}\]

\[C_t(z) = z^1 + S_t^a z^2 1_{z^2>0} + S_t^b z^2 1_{z^2<0}, \text{ bid – ask}\]

and the model with fixed cost (non-convex cost):

\[C_t(z) = z^1 - (-z^2 S_t^b - c_t)^+ 1_{z^2<0} + (z^2 S_t^a - c_t) 1_{z^2>0}\]
We denote by $\mathcal{R}_t(\xi)$ the set of all portfolio processes starting at time $t \leq T$ that replicates $\xi$ at the terminal date $T$:

$$\mathcal{R}_t(\xi) := \{(V_s)_{s=t}^T, -\Delta V_s \in L^0(G_s, \mathcal{F}_s), \forall s \geq t + 1, V_T = \xi\}.$$ 

The set of replicating prices of $\xi$ at time $t$ is

$$\mathcal{P}_t(\xi) := \left\{ V_t = (V_t^1, V_t^{(2)}) : (V_s)_{s=t}^T \in \mathcal{R}_t(\xi) \right\}.$$ 

The infimum replicating cost is then defined as:

$$c_t(\xi) := \text{ess inf}_{\mathcal{F}_t} \{ C_t(V_t), V_t \in \mathcal{P}_t(\xi) \}.$$ 

We are interested in $c_0(\xi)$.
Definition (Conditional essential infimum)

Let $\mathcal{H}$ and $\mathcal{F}$ be complete $\sigma$-algebras such that $\mathcal{H} \subseteq \mathcal{F}$ and let $\Gamma = (\gamma_i)_{i \in I}$ be a family of real-valued $\mathcal{F}$-measurable random variables. There exists a unique (up to a $P$-negligible set) random variables $\gamma_\mathcal{H} \in L^0(\mathbb{R}, \mathcal{H})$, denoted by $\text{ess inf}_\mathcal{H} \Gamma$, which satisfies the following properties

1) For every $i \in I$, $\gamma_\mathcal{H} \leq \gamma_i$ a.s.

2) If $\zeta \in L^0(\mathbb{R}, \mathcal{H})$ satisfies $\zeta \leq \gamma_i$ a.s. for all $i \in I$, then $\zeta \leq \gamma_\mathcal{H}$ a.s.
In classical approach, using dual characterization of $c_0(\xi)$ via Consistent Price Systems.

Numerical algorithm is available only for finite probability space.
Important observation: \( L_T(-\Delta V_T) \geq 0 \) is equivalent to:

\[
V^1_{T-1} \geq \xi^1 + C_T((0, \xi^{(2)} - V^{(2)}_{T-1})),
\]

\[\iff V^1_{T-1} \geq \text{ess sup}_{\mathcal{F}_{T-1}} \left( \xi^1 + C_T((0, \xi^{(2)} - V^{(2)}_{T-1})) \right), \text{ a.s.}\]

We continue,

\[
V^1_t \geq \text{ess sup}_{\mathcal{F}_t} \left( \xi^1 + \sum_{s=t+1}^{T} C_s(0, V^{(2)}_s - V^{(2)}_{s-1}) \right).
\]
Computable prices in model with transaction costs

We set:

\[ \Pi^T_u (\xi_{u-1}, \xi) := \{ \xi^{(2)}_{u-1} \} \times \Pi^{T-1}_{s=u} L^0(\mathbb{R}^{d-1}, \mathcal{F}_s) \times \{ \xi^{(2)} \} \]

We define \( \gamma^\xi_t (V_{t-1}) \) as:

\[
\gamma^\xi_t (V_{t-1}) := \underset{V^{(2)} \in \Pi^T_t (V_{t-1}, \xi)}{\operatorname{ess inf}} \underset{V_t \in L^0(\mathbb{R}^d, \mathcal{F}_t)}{\operatorname{ess sup}} \left( \xi^1 + \sum_{s=t}^T C_s(0, V_{s}^{(2)} - V_{s-1}^{(2)}) \right).
\]

**Theorem (Dynamic Programming Principle)**

*For any \( 0 \leq t \leq T - 1 \) and \( V_{t-1} \in L^0(\mathbb{R}^d, \mathcal{F}_{t-1}) \), we have*

\[
\gamma^\xi_t (V_{t-1}) = \underset{V_t \in L^0(\mathbb{R}^d, \mathcal{F}_t)}{\operatorname{ess inf}} \underset{V_t \in L^0(\mathbb{R}^d, \mathcal{F}_t)}{\operatorname{ess sup}} \left( C_t(0, V_{t}^{(2)} - V_{t-1}^{(2)}) + \gamma^\xi_{t+1} (V_t) \right).
\]

*In particular, \( \gamma^\xi_t (0) = c_0(\xi) \).*
Usually, we have $\gamma_t^\xi (V_{t-1}) = \gamma_t^\xi (S_t, V_{t-1})$. We consider a weak NA condition:

**Definition (Absence of Immediate Profit : AIP)**

For any $t \leq T - 1$, the minimal cost to hedge zero payoff is identical to 0 a.s. : $c_t(0) = 0$.

**Definition (Strong Absence of Immediate Profit : SAIP)**

For any $t \leq T - 1$, the minimal cost to hedge zero payoff is identical to 0 a.s. : $c_t(0) = 0$ (AIP holds). Moreover, $c_t(0)$ is attained only at the zero strategy.
Computable prices in model with transaction costs

**Theorem (ω-wise formulation of DPP)**

*Under SAIP, the DPP is computable ω-wise as:

\[
\gamma_{t}^{\xi}(S_t, V_{t-1}) = \inf_{y \in \mathbb{R}^d} \left( C_t(S_t, (0, y^{(2)} - V_{t-1}^{(2)})) + \sup_{s \in \phi_t(S_t)} \gamma_{t+1}^{\xi}(s, y) \right),
\]

where \( \phi_t(S_t) = \text{supp}_{\mathcal{F}_t} S_{t+1} \). Also, the infimum hedging cost of \( \xi \) at any time \( t \) is reached.*

Main idea: maintain the lower semicontinuity of \( \gamma_{t}^{\xi} \).
Theorem (Reachability set)

Under SAIP, the DPP is computable $\omega$-wise as:

$$
\gamma_t^\xi(S_t, V_{t-1}) = \inf_{y \in K_t(S_t, V_{t-1})} \left( C_t(S_t, (0, y^{(2)} - V_{t-1}^{(2)})) + \sup_{s \in \phi_t(S_t)} \gamma_{t+1}^\xi(s, y) \right),
$$

for some compact set-valued mapping $K_t : \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{R}^d$. 
We consider Binomial model $\text{supp}_t S_{t+1} = \{ k_t^d S_t, k_t^u S_t \}$, where $k_t^d, k_t^u \in \mathbb{R}_+$ and the cost function :

$$C_t(S_t, v) = v^1 + S_t^a v^2 1_{v^2 \geq 0} + S_t^b v^2 1_{v^2 \leq 0}.$$ 

We want to hedge the payoff $\xi = ((K - S_T)^+, 0) \in \mathbb{R}^2$. 

We suppose that $S_t^a = S_t(1 + \epsilon_t), S_t^b = S_t(1 - \epsilon_t), \epsilon_t \in \mathbb{R}_+$. 
Bid-Ask spread

**Theorem**

AIP\(T_{-1}\) holds if and only if:

\[k^d_{T-1} \leq \frac{1 + \epsilon_{T-1}}{1 - \epsilon_T} \text{ and } k^u_{T-1} \geq \frac{1 - \epsilon_{T-1}}{1 + \epsilon_T}.\]

Moreover, SAIP\(T_{-1}\) holds if and only if the above inequalities are strict. Moreover, suppose that \(1 + \epsilon_{T-1} \leq (1 + \epsilon_T)k^u_{T-1}\) and \(1 - \epsilon_{T-1} \geq (1 - \epsilon_T)k^d_{T-1}\), AIP\(T_{-2}\) holds if and only if:

\[k^d_{T-2} \leq \frac{1 + \epsilon_{T-2}}{1 + \epsilon_{T-1}} \text{ and } k^u_{T-2} \geq \frac{1 - \epsilon_{T-2}}{(1 - \epsilon_T)k^d_{T-1}}.\]

Moreover, SAIP\(T_{-2}\) holds if and only if the above inequalities are strict.
Bid-Ask spread

We suppose $T = 2$, the proportional cost coefficients $\epsilon_1 = \epsilon_2 = 0.02$. We assume that SAIP condition holds and choose $k_d^2 = 0.9$, $k_u^2 = 1.1$, $k_d^1 = 0.9$, $k_u^1 = 1.2$.

Figure – Price of Put option
Consider the cost function:

\[ C_t(z) = z^1 - (-z^2 S_t^b - c_t)^+ 1_{z^2 < 0} + (z^2 S_t^a - c_t) 1_{z^2 > 0} \]

We suppose that the market defined by \( C_t \) with \( c_t = 0 \) satisfies SAIP. We called this condition Robust SAIP or RSAIP.
Fixed cost

We use the same parameters as Bid-Ask spread and we consider fixed costs $c_1 = c_2 = 0.8$.

Figure – Price of put option with fixed costs.

Thank you for your attention!

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