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An alternative pricing method for financial models with transaction costs

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Computable prices in model with transaction costs

We consider a portfolio process : $(V_t)_{t=-1}^T$ where $V_{-1} \in \mathbf{R}e_1$ is the initial endowment expressed in cash.

The solvency set $\mathbf{G}_t : \Omega \twoheadrightarrow \mathbf{R}^d$ is $\mathcal{F}_t$-measurable random closed set,i.e.

$$\mathrm{Graph}(\mathbf{G}_t) = \{(\omega, x) : \ x \in \mathbf{G}_t(\omega)\} \in \mathcal{F}_t \otimes \mathcal{B}(\mathbf{R}^d)$$

The cost process $\mathbf{C} = (\mathbf{C}_t)_{t=0}^T$ associated to $\mathbf{G}$ is defined as :

$$\mathbf{C}_t(z) = \inf\{\alpha \in \mathbf{R} : \ \ \alpha e_1 - z \in \mathbf{G}_t\} = \min\{\alpha \in \mathbf{R} : \ \ \alpha e_1 - z \in \mathbf{G}_t\}$$

$\mathbf{C}_t$ is lower semicontinuous and is allowed to be non-convex.
Similarly, we may define the liquidation value process $L = (L_t)_{t=0}^T$:

$$L_t(z) := \sup\{\alpha \in \mathbf{R} : z - \alpha e_1 \in \mathbf{G}_t\}, \quad z \in \mathbf{R}^d.$$

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A portfolio process is a stochastic process $(V_t)_{t=-1}^{T}$ satisfying :

$$\Delta V_t = V_t - V_{t-1} \in -\mathbf{G}_t, \ a.s., \quad t = 0, \cdots, T.$$

or equivalently, $L_t(-\Delta V_t) \geq 0$ a.s.

Some examples :

$$\mathbf{C}_t(z) = zS_t, \text{ no transaction cost}$$
$$\mathbf{C}_t(z) = z^1 + S_t^a z^2 1_{z^2>0} + S_t^b z^2 1_{z^2<0}, \text{ bid} - \text{ask}$$

and the model with fixed cost (non-convex cost) :

$$\mathbf{C}_t(z) = z^1 - \left(-z^2 S_t^b - c_t\right)^+ 1_{z^2<0} + \left(z^2 S_t^a - c_t\right) 1_{z^2>0}$$

Computable prices in model with transaction costs

We denote by $\mathcal{R}_t(\xi)$ the set of all portfolio processes starting at time $t \leq T$ that replicates ξ at the terminal date T :

$$\mathcal{R}_t(\xi) := \left\{(V_s)_{s=t}^T, -\Delta V_s \in L^0(\mathbf{G}_s, \mathcal{F}_s), \, \forall s \geq t+1, V_T = \xi\right\}.$$

The set of replicating prices of ξ at time t is

$$\mathcal{P}_t(\xi) := \left\{V_t = (V_t^1, V_t^{(2)}) : (V_s)_{s=t}^T \in \mathcal{R}_t(\xi)\right\}.$$

The infimum replicating cost is then defined as :

$$c_t(\xi) := \text{ess}\inf{}_{\mathcal{F}_t} \left\{\mathbf{C}_t(V_t), \; V_t \in \mathcal{P}_t(\xi)\right\}.$$

We are interested in $c_0(\xi)$

Definition (Conditional essential infimum)

Let $\mathcal{H}$ and $\mathcal{F}$ be complet σ-algebras such that $\mathcal{H} \subseteq \mathcal{F}$ and let $\Gamma = (\gamma_i)_{i \in I}$ be a family of real-valued $\mathcal{F}$-measurable random variables. There exists a unique (up to a P-negligible set) random variables $\gamma_{\mathcal{H}} \in L^0(\overline{\mathbf{R}}, \mathcal{H})$, denoted by $\operatorname{ess\,inf}_{\mathcal{H}} \Gamma$, which satisfies the following properties

1) *For every $i \in I$, $\gamma_{\mathcal{H}} \leq \gamma_i$ a.s.*
2) *If $\zeta \in L^0(\overline{\mathbf{R}}, \mathcal{H})$ satisfies $\zeta \leq \gamma_i$ a.s. for all $i \in I$, then $\zeta \leq \gamma_{\mathcal{H}}$ a.s.*

Computable prices in model with transaction costs

In classical approach, using dual characterization of $c_0(\xi)$ via Consistent Price Systems.

Numerical algorithm is available only for finite probability space.

Computable prices in model with transaction costs

Important observation : $L_T(-\Delta V_T) \geq 0$ is equivalent to :

$$
\begin{aligned}
& V^1_{T-1} \geq \xi^1 + \mathbf{C}_T((0, \xi^{(2)} - V^{(2)}_{T-1})), \\
\Longleftrightarrow & V^1_{T-1} \geq \text{ess sup}_{\mathcal{F}_{T-1}} \left(\xi^1 + \mathbf{C}_T((0, \xi^{(2)} - V^{(2)}_{T-1})) \right), \ a.s.
\end{aligned}
$$

We continue,

$$
V^1_t \geq \text{ess sup}_{\mathcal{F}_t} \left(\xi^1 + \sum_{s=t+1}^{T} \mathbf{C}_s(0, V^{(2)}_s - V^{(2)}_{s-1}) \right).
$$

Computable prices in model with transaction costs

We set :

$$\Pi_u^T(\xi_{u-1},\xi) := \{\xi_{u-1}^{(2)}\} \times \Pi_{s=u}^{T-1} L^0(\mathbf{R}^{d-1},\mathcal{F}_s) \times \{\xi^{(2)}\}$$

We define $\gamma_t^{\xi}(V_{t-1})$ as :

$$\gamma_t^{\xi}(V_{t-1}) := \underset{V^{(2)} \in \Pi_t^T(V_{t-1},\xi)}{\text{ess inf}_{\mathcal{F}_t}} \ \text{ess sup}_{\mathcal{F}_t} \left(\xi^1 + \sum_{s=t}^{T} \mathbf{C}_s(0, V_s^{(2)} - V_{s-1}^{(2)}) \right).$$

Theorem (Dynamic Programming Principle)

For any $0 \leq t \leq T-1$ *and* $V_{t-1} \in L^0(\mathbf{R}^d, \mathcal{F}_{t-1})$, *we have*

$$\gamma_t^{\xi}(V_{t-1}) = \underset{V_t \in L^0(\mathbf{R}^d,\mathcal{F}_t)}{\text{ess inf}_{\mathcal{F}_t}} \ \text{ess sup}_{\mathcal{F}_t} \left(\mathbf{C}_t(0, V_t^{(2)} - V_{t-1}^{(2)}) + \gamma_{t+1}^{\xi}(V_t) \right).$$

In particular, $\gamma_t^{\xi}(0) = c_0(\xi)$.

Computable prices in model with transaction costs

Usually, we have $\gamma_t^\xi(V_{t-1}) = \gamma_t^\xi(S_t, V_{t-1})$. We consider a weak NA condition :

Definition (Absence of Immediate Profit : AIP)

For any $t \leq T-1$, the minimal cost to hedge zero payoff is identical to 0 a.s. : $c_t(0) = 0$.

Definition (Strong Absence of Immediate Profit : SAIP)

For any $t \leq T-1$, the minimal cost to hedge zero payoff is identical to 0 a.s. : $c_t(0) = 0$ (AIP holds). Moreover, $c_t(0)$ is attained only at the zero strategy.

Computable prices in model with transaction costs

Theorem (ω-wise formulation of DPP)

Under SAIP, the DPP is computable ω-wise as :

$$\gamma_t^{\xi}(S_t, V_{t-1}) = \inf_{y \in \mathbf{R}^d} \left(\mathbf{C}_t(S_t, (0, y^{(2)} - V_{t-1}^{(2)})) + \sup_{s \in \phi_t(S_t)} \gamma_{t+1}^{\xi}(s, y) \right),$$

where $\phi_t(S_t) = \mathrm{supp}_{\mathcal{F}_t} S_{t+1}$. Also, the infimum hedging cost of ξ at any time t is reached.

Main idea : maintain the lower semicontinuity of γ_t^{ξ}.

Computable prices in model with transaction costs

Theorem (Reachability set)

Under SAIP, the DPP is computable ω-wise as :

$$\gamma_t^{\xi}(S_t, V_{t-1}) = \inf_{y \in K_t(S_t, V_{t-1})} \Bigg(\mathbf{C}_t(S_t, (0, y^{(2)} - V_{t-1}^{(2)})) + \sup_{s \in \phi_t(S_t)} \gamma_{t+1}^{\xi}(s, y) \Bigg),$$

for some compact set-valued mapping $K_t : \mathbf{R}^d \times \mathbf{R}^d \twoheadrightarrow \mathbf{R}^d$.

Bid-Ask spread

We consider Binomial model $\mathrm{supp}_{\mathcal{F}_t} S_{t+1} = \left\{ k_t^d S_t, k_t^u S_t \right\}$, where $k_t^d, k_t^u \in \mathbf{R}_+$ and the cost function :

$$\mathbf{C}_t(S_t, v) = v^1 + S_t^a v^2 1_{v^2 \geq 0} + S_t^b v^2 1_{v^2 \leq 0}.$$

We want to hedge the payoff $\xi = ((K - S_T)^+, 0) \in \mathbf{R}^2$.

We suppose that $S_t^a = S_t(1 + \epsilon_t), S_t^b = S_t(1 - \epsilon_t)$, $\epsilon_t \in \mathbf{R}_+$.

Theorem

AIP_{T-1} *holds if and only if :*

$$k^d_{T-1} \leq \frac{1+\epsilon_{T-1}}{1-\epsilon_T} \text{ and } k^u_{T-1} \geq \frac{1-\epsilon_{T-1}}{1+\epsilon_T}.$$

Moreover, SAIP_{T-1} *holds if and only if the above inequalities are strict. Moreover, suppose that* $1+\epsilon_{T-1} \leq (1+\epsilon_T)k^u_{T-1}$ *and* $1-\epsilon_{T-1} \geq (1-\epsilon_T)k^d_{T-1}$, AIP_{T-2} *holds if and only if :*

$$k^d_{T-2} \leq \frac{1+\epsilon_{T-2}}{1+\epsilon_{T-1}} \text{ and } k^u_{T-2} \geq \frac{1-\epsilon_{T-2}}{(1-\epsilon_T)k^d_{T-1}}.$$

Moreover, SAIP_{T-2} *holds if and only if the above inequalities are strict*

We suppose $T = 2$, the proportional cost coefficients $\epsilon_1 = \epsilon_2 = 0.02$. We assume that SAIP condition holds and choose $k_2^d = 0.9, k_2^u = 1.1,\ k_1^d = 0.9, k_1^u = 1.2$.

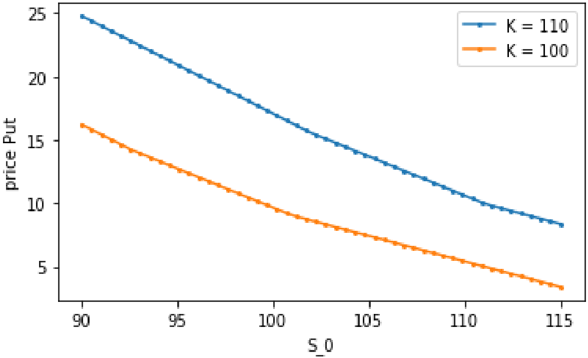

Figure – Price of Put option

Fixed cost

Consider the cost function :

$$\mathbf{C}_t(z) = z^1 - \left(-z^2 S_t^b - c_t\right)^+ 1_{z^2<0} + \left(z^2 S_t^a - c_t\right) 1_{z^2>0}$$

We suppose that the market defined by $\mathbf{C}_t$ with $c_t = 0$ satisfies SAIP. We called this condition Robust SAIP or RSAIP.

Fixed cost

We use the same parameters as Bid-Ask spread and we consider fixed costs $c_1 = c_2 = 0.8$.

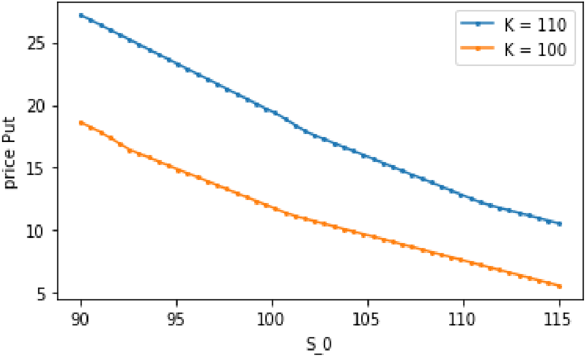

Figure – Price of put option with fixed costs.

References

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Thank you for your attention !

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