Barycentres de séries temporelles : une nouvelle approche basée sur la méthode de la signature

Journées MAS (Modélisation Aléatoire et Statistique) 2022 de la Société de Mathématiques Appliquées et Industrielles

Raphaël Mignot

Institut Elie Cartan de Lorraine
Université de Lorraine
raphael.mignot@univ-lorraine.fr

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Joint work with M. Clausel, K. Usevich, G. Oppenheim, L. Coutin, A. Lejay

Background

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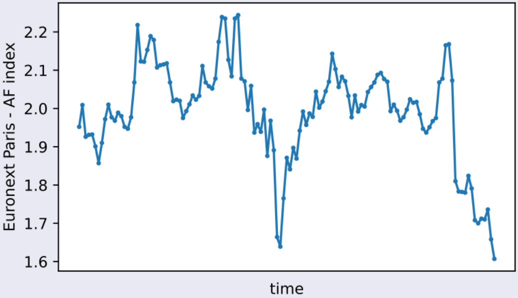
Time series



→ sequential data appears in many contexts!

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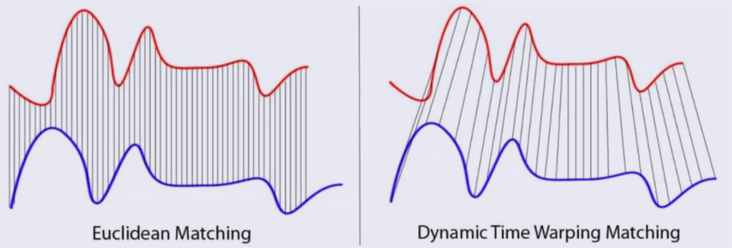
Time series

- Similarity measure between two time series ?



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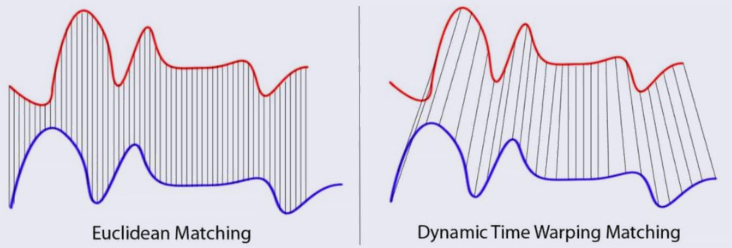

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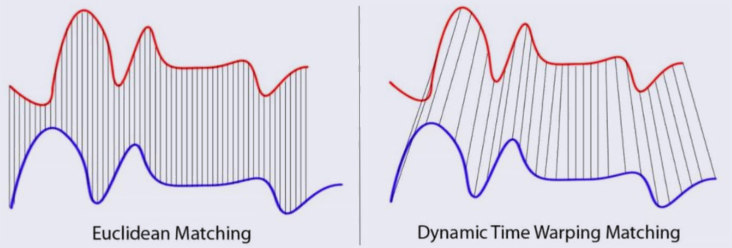

- Euclidean → not accurate for many tasks
- Dynamic Time Warping (DTW) → relevant, versatile

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$\rightarrow$ How to average a set of time series?

What has been done?

- Pairwise averaging (with eg. euclidean distance)
- Averaging method using DTW[1]
- Averaging methods on Lie groups

[1] Petitjean, F., Ketterlin, A., & Gançarski, P. (2011). A global averaging method for dynamic time warping, with applications to clustering. Pattern recognition, 44(3), 678-693. [PKG11]

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with $\otimes$ the tensor product.

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  with $\otimes$ the tensor product.
- Signature of X is the infinite collection of signatures of all orders:

$$\mathbf{S}_{[0,1]}(X) = (1, S^{(1)}_{[0,1]}(X), S^{(2)}_{[0,1]}(X), S^{(3)}_{[0,1]}(X), \dots)$$

Chevyrev, I., & Kormilitzin, A. (2016). A primer on the signature method in machine learning. arXiv preprint arXiv:1603.03788. [CK16]

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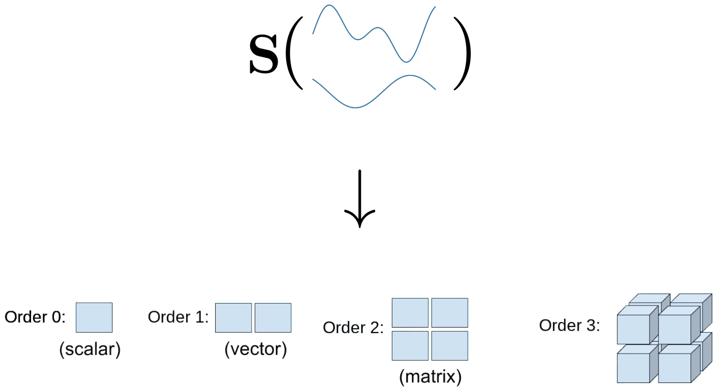

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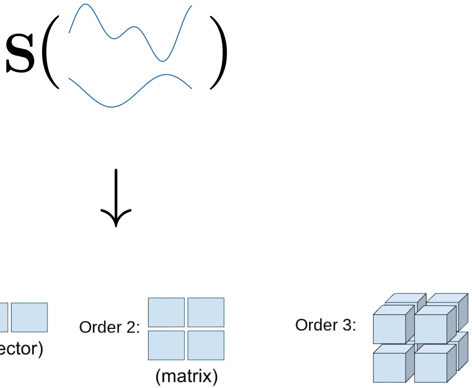

- Time series: $X \in \mathbb{R}^{2 \times 100}$

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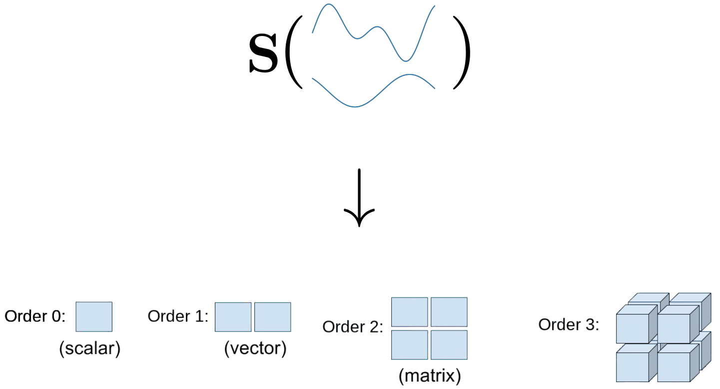

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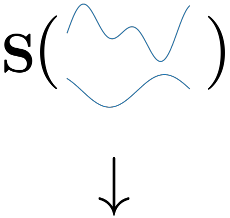

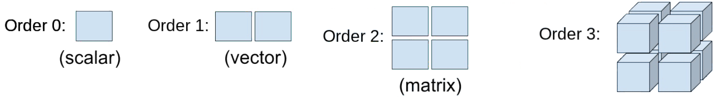

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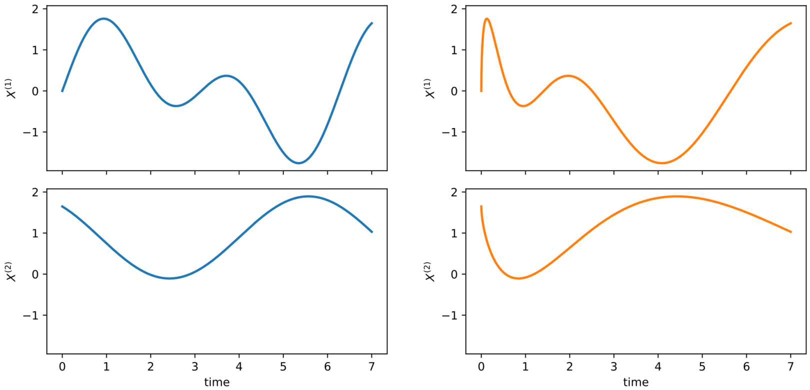

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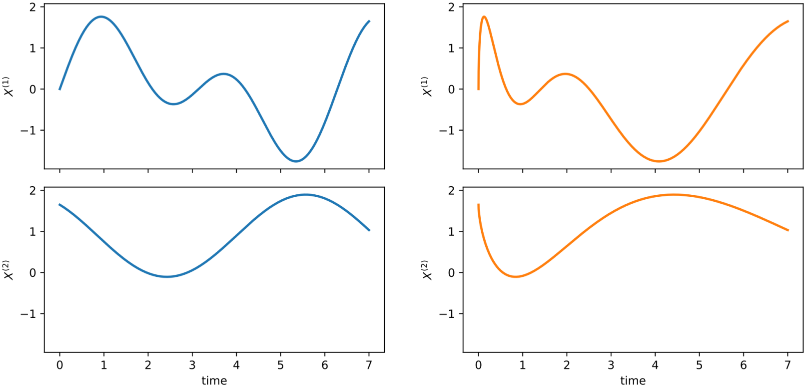


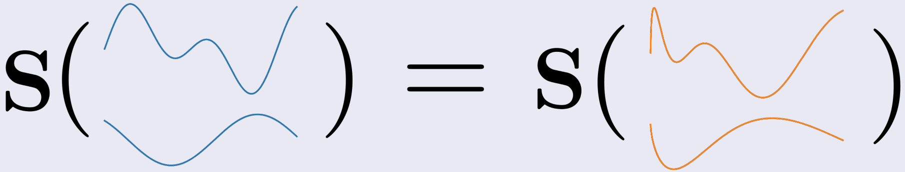
→ Same paths, but different time parametrizations!

The signature transform

Fundamental property #1 of the signature transform

- **Intrinsic characterization** of the path, ignoring translation and time reparametrization: let φ be a reparametrisation

$$\mathbf{S}_{[a,b]}(X_{\varphi(.)}) = \mathbf{S}_{[\varphi(a),\varphi(b)]}(X)$$



The signature transform

Fundamental property #2 of the signature transform

- The space of signatures is a non compact Lie group under $\otimes$ operation[1].

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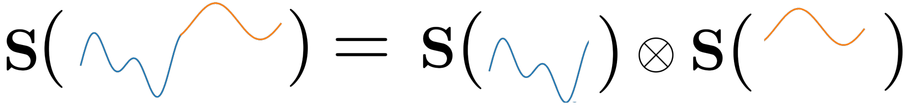
The signature transform

Fundamental property #2 of the signature transform

- The space of signatures is a non compact Lie group under $\otimes$ operation[1].
- This $\otimes$ operation is related to the concatenation of two paths through the so-called Chen relation: let $0 \leq u \leq 1$

$$\mathbf{S}_{[0,1]}(X \star Y) = \mathbf{S}_{[0,u]}(X) \otimes \mathbf{S}_{[u,1]}(Y)$$



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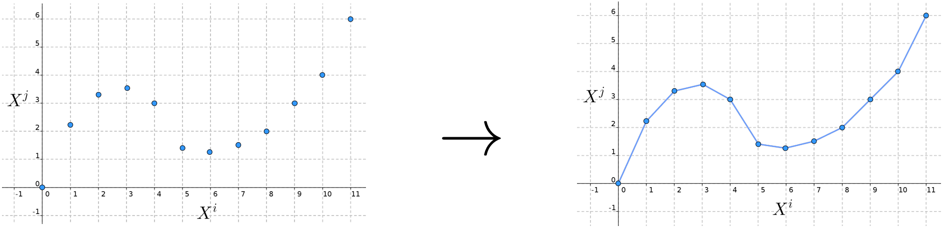

$\rightarrow$ linear interpolation

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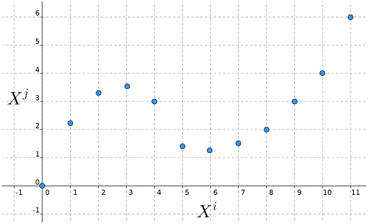
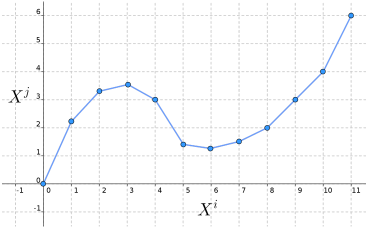


$\longrightarrow$



$\rightarrow$ linear interpolation

$\rightarrow$ why? easy to compute because of Chen relation

Averaging time series

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$\rightarrow$ all three are based on Fundamental prop. #2: signature $\in$ Lie group

Averaging the signature

Approach 1: the Log Euclidean barycenter

Let $x_1, \ldots, x_n$ be n multidimensional time series
Let $X_1, \ldots, X_n$ be their linear interpolations
Denote $\mathbb{X}_i = \mathbf{S}_{[0,1]}(X_i)$.

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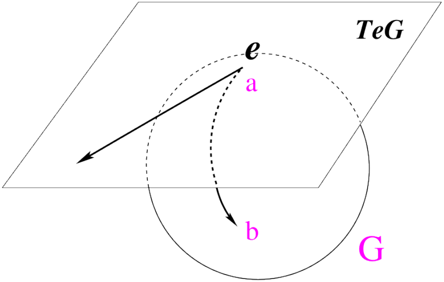

Figure: Lie Group G and its Lie algebra $T_e G$.

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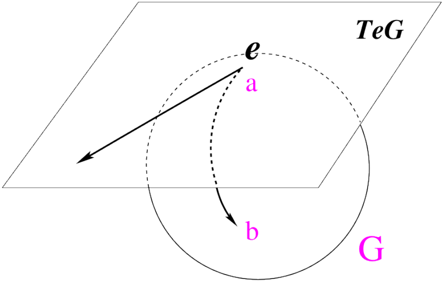

Figure: Lie Group G and its Lie algebra $T_e G$.

Definition (Approach 1)

The **Log Euclidean barycenter** is

$$\bar{\mathbb{X}}_{LE} = \exp\left(\frac{1}{n}\sum_{i=1}^{n}\log(\mathbb{X}_i)\right).$$

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Pennec & Arsigny (2013). Exponential barycenters of the canonical Cartan connection and invariant means on Lie groups. [PA13, Algorithm 1 & Corollary 5]

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The barycenter is the path $\bar{X} \in \mathbb{R}^{D \times L}$ such that

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- Distance d: euclidean, signature distance.
- Optimization: gradient descent.

Application: clustering time series

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K-means clustering

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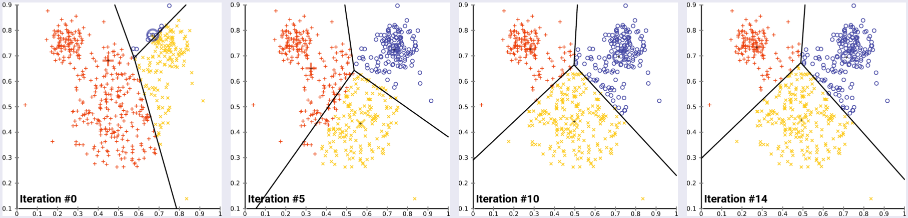
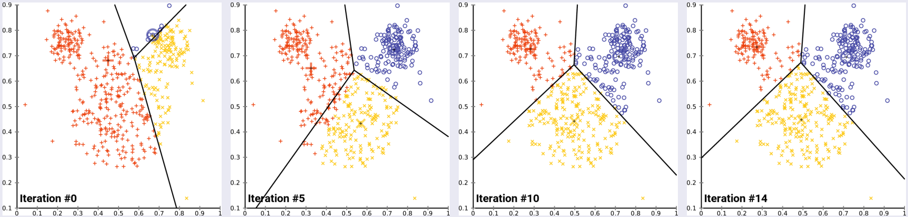


Application

K-means clustering

- **Goal:** cluster n observations $(x_i)_{i=1,\dots,n}$ into K groups.
- Two parameters to choose:
  1. similarity measure
  2. averaging method

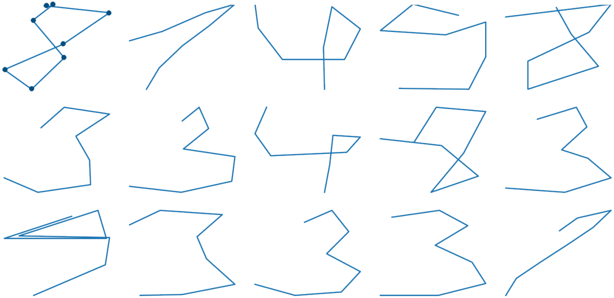


Application

Framework

- Dataset: PenDigits[1]
- 11000 bidimensional time series of length 8.

```
data.shape = (11000, 8, 2)
```



[1] Bagnall et al. 2018. The UEA multivariate time series classification archive.

Application

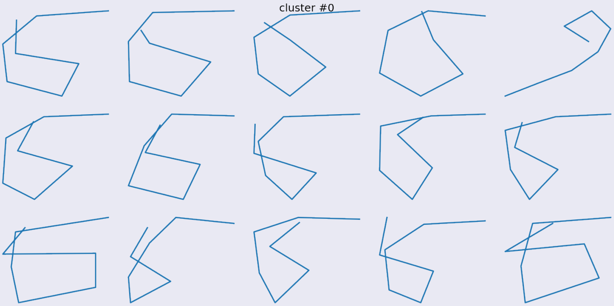


Figure: K-means' cluster #0 with averaging Approach 3.

Application: results

Table: K-means parameters used for benchmark. Pink indicates the use of the signature transform.

Averaging method	**Similarity measure**
Euclidean barycenter	euclidean
DTW Barycenter Averaging[1]	DTW
Approach 1: log-euclidean	euclidean
Approach 2: group exponential barycenter	euclidean
Approach 3: path space optimization	euclidean

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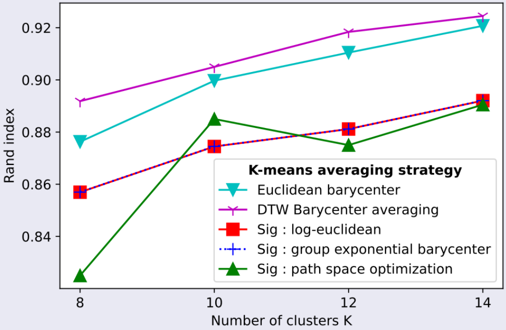
Rand Index



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Regarding the averaging approaches

- Extend other methods to the time series framework.
- How to represent a barycenter?

Thank you!

Bibliography

[CK16] Ilya Chevyrev and Andrey Kormilitzin. A primer on the signature method in machine learning. *arXiv:1603.03788 [cs, stat]*, Mar 2016. arXiv: 1603.03788.

[PA13] Xavier Pennec and Vincent Arsigny. *Exponential Barycenters of the Canonical Cartan Connection and Invariant Means on Lie Groups*, page 123–166. Springer Berlin Heidelberg, 2013.

[PKG11] François Petitjean, Alain Ketterlin, and Pierre Gançarski. A global averaging method for dynamic time warping, with applications to clustering. *Pattern Recognition*, 44(3):678–693, Mar 2011.