Local scaling limits of Lévy driven fractional random fields

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joint work with
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Plan of this talk[1]

1. Introduction
2. Lévy driven fractional random fields (RFs). Examples
3. Main results
4. Extensions

[1] Pilipauskaitė, V., Surgailis, D. (2022+). Local scaling limits of Lévy driven fractional random fields. To appear in Bernoulli.

1. Introduction

Let $X = \{X(\boldsymbol{t})\}_{\boldsymbol{t}\in\mathbb{R}^2}$ be a RF. At $\boldsymbol{t}_0 = (t_1^0, t_2^0) \in \mathbb{R}^2$ for $\boldsymbol{t} = (t_1, t_2) \in \mathbb{R}_+^2$,

- ordinary increment of X is defined as

$$X(\boldsymbol{t}_0 + \boldsymbol{t}) - X(\boldsymbol{t}_0),$$

- rectangular increment of X is defined as

$$\begin{aligned} X(t_1^0 + t_1, t_2^0 + t_2) - X(t_1^0 + t_1, t_2^0) - X(t_1^0, t_2^0 + t_2) + X(t_1^0, t_2^0) \\ = \Delta_{t_1}^{(1)} \Delta_{t_2}^{(2)} X(\boldsymbol{t}_0), \end{aligned}$$

  where $\Delta_t^{(i)} X(\boldsymbol{t}_0) = X(\boldsymbol{t}_0 + t\boldsymbol{e}_i) - X(\boldsymbol{t}_0)$, $i = 1, 2$, with $\boldsymbol{e}_1 = (1, 0)$, $\boldsymbol{e}_2 = (0, 1)$ and $t \in \mathbb{R}_+$.

We study the scaling limit of increments of X at $\boldsymbol{t}_0$ for $\gamma > 0$ as $\lambda \downarrow 0$:

$$\left\{c_{\lambda,\gamma}^{-1}(X(t_1^0 + \lambda t_1, t_2^0 + \lambda^{\gamma} t_2) - X(t_1^0, t_2^0)\right\}_{\boldsymbol{t}\in\mathbb{R}_+^2} \overset{\text{fdd}}{\to} \left\{T_{\gamma}(\boldsymbol{t})\right\}_{\boldsymbol{t}\in\mathbb{R}_+^2}, \qquad (1)$$

$$\left\{C_{\lambda,\gamma}^{-1}\Delta_{\lambda t_1}^{(1)}\Delta_{\lambda^{\gamma} t_2}^{(2)} X(\boldsymbol{t}_0)\right\}_{\boldsymbol{t}\in\mathbb{R}_+^2} \overset{\text{fdd}}{\to} \left\{R_{\gamma}(\boldsymbol{t})\right\}_{\boldsymbol{t}\in\mathbb{R}_+^2}, \qquad (2)$$

where

- $\overset{\text{fdd}}{\to}$ denotes the weak convergence of finite-dimensional distributions,
- $c_{\lambda,\gamma}, C_{\lambda,\gamma} > 0$ are normalizing constants,
- T_γ, R_γ are non-trivial RFs.

We call T_γ, R_γ respectively the γ-tangent RF, γ-rectangent RF of X at $\boldsymbol{t}_0$.

- Definition and some properties of T_1[2].
- Local asymptotic self-similarity of X[3,4] $\Longrightarrow$ Existence of T_1.

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[2] Falconer, K.J. (2002). Tangent fields and the local structure of random fields. J. Theoret. Probab. 15, 731–750.

[3] Benassi A., Roux D., Jaffard S. (1997). Elliptic gaussian random processes. Rev. Mat. Iberoam. 13, 19–90.

[4] Benassi, A., Cohen, S., Istas, J. (2004). On roughness indices for fractional fields. Bernoulli 10, 357–373.

- Is there a scaling transition[5,6,7,8,9,10,11]?
  For some classes of long-range/negatively dependent RFs X, for all $\gamma > 0$, as $\lambda \to \infty$,

$$\left\{ C_{\lambda,\gamma}^{-1} \int_{[0,\lambda t_1]\times[0,\lambda^{\gamma} t_2]} X(\boldsymbol{s})\mathrm{d}\boldsymbol{s} \right\}_{\boldsymbol{t}\in\mathbb{R}_+^2} \stackrel{\text{fdd}}{\to} \{S_\gamma(\boldsymbol{t})\}_{\boldsymbol{t}\in\mathbb{R}_+^2},$$

  where the limits undergo a transition at some $\gamma_0 > 0$, i.e.

$$S_\gamma = \begin{cases} S_+, & \gamma > \gamma_0, \\ S_0, & \gamma = \gamma_0, \\ S_-, & \gamma < \gamma_0, \end{cases} \quad \text{and} \quad S_+ \stackrel{\text{fdd}}{\neq} cS_- \text{ for all } c > 0.$$

- Applications to infill statistics for RFs.

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[5] Puplinskaitė, D., Surgailis, D. (2015). Stochastic Process. Appl. 125, 2256–2271.
[6] ——— (2016). Bernoulli 22, 2401–2441.
[7] Pilipauskaitė, V., Surgailis, D. (2016). J. Appl. Probab. 53, 857–879.
[8] ——— (2017). Stochastic Process. Appl. 127, 2751–2779.
[9] ——— (2021). In: M.E. Vares et al. (Eds.) In and Out of Equilibrium 3: Celebrating Vladas Sidoravicius. Progress in Probability, pp. 683–710. Birkhäuser.
[10] Surgailis, D. (2020). Stochastic Process. Appl. 130, 7518–7546.
[11] Biermé, H., Durieu, O., Wang, Y. (2017). Ann. Appl. Probab. 27, 1190–1234.

2. Lévy driven fractional RFs. Examples

Consider

$$X(\boldsymbol{t}) = \int_{\mathbb{R}^2} g(\boldsymbol{t}, \boldsymbol{u}) M(\mathrm{d}\boldsymbol{u}), \quad \boldsymbol{t} \in \mathbb{R}^2,$$

where

- $g : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ is a measurable deterministic function,
- $\{M(B)\}_{B \in \mathcal{B}_b(\mathbb{R}^2)}$ is an infinitely divisible random measure, i.e.
  - if $\{B_i\}_i$ is a sequence of disjoint sets in $\mathcal{B}_b(\mathbb{R}^2)$, then $\{M(B_i)\}_i$ is a sequence of independent random variables, and, moreover, if $\cup_i B_i \in \mathcal{B}_b(\mathbb{R}^2)$, then $M(\cup_i B_i) = \sum_i M(B_i)$ a.s.,
  - $M(B)$ has infinitely divisible distribution, $B \in \mathcal{B}_b(\mathbb{R}^2)$.

Let $0 < \alpha \leq 2$. Assume

(M)$_\alpha$ for every $B \in \mathcal{B}_b(\mathbb{R}^2)$, the characteristic function of $M(B)$ is

$$\mathbb{E}\mathrm{e}^{\mathrm{i}\theta M(B)} = \exp\left\{ \mathrm{Leb}(B)\left(-\frac{1}{2}\sigma^2\theta^2 + \int_{\mathbb{R}} (\mathrm{e}^{\mathrm{i}\theta y} - 1 - \mathrm{i}\theta\tau_\alpha(y))\nu(\mathrm{d}y) \right) \right\}, \quad \theta \in \mathbb{R},$$

where

$\tau_\alpha(y) = 0$ if $\alpha < 1$, $\tau_\alpha(y) = y\mathbf{1}(|y| \leq 1)$ if $\alpha = 1$, $\tau_\alpha(y) = y$ if $\alpha > 1$,

and $\sigma^2 \geq 0$, ν is a Lévy measure on $\mathbb{R}$ such that

(i) if $\alpha < 2$, then

$$\sigma = 0, \qquad \lim_{y \downarrow 0} y^\alpha \nu((y, \infty)) = c_+, \qquad \lim_{y \downarrow 0} y^\alpha \nu((-\infty, -y)) = c_-$$

for some $c_\pm \geq 0$, $c_+ + c_- > 0$ and $\sup_{y>0} y^\alpha \nu(\{u \in \mathbb{R} : |u| > y\}) < \infty$, moreover, if $\alpha = 1$, then ν is symmetric,

(ii) if $\alpha = 2$, then $\sigma > 0$, $\int_{\mathbb{R}} y^2 \nu(\mathrm{d}y) < \infty$.

Then for all $\gamma > 0$, as $\lambda \downarrow 0$,

$$\lambda^{-\frac{1}{\alpha}(1+\gamma)} M([0, \lambda] \times [0, \lambda^\gamma]) \xrightarrow{\mathrm{d}} W_\alpha([0, 1] \times [0, 1])$$

where W_α is α-stable random measure with control measure σ_0^α Leb and skewness intensity $\beta_0 \in [-1, 1]$.

Let $0 < \alpha \leq 2$ be the same. Assume

(G)$_\alpha$ $g(\boldsymbol{t}, \cdot) \in L^\alpha(\mathbb{R}^2)$, moreover,

$$\begin{aligned} g(\boldsymbol{t}, \boldsymbol{u}) = g(t_1 - u_1, t_2 - u_2) &- g_2(t_1 - u_1, -u_2) \\ &- g_1(-u_1, t_2 - u_2) + g_{12}(-u_1, -u_2), \quad \boldsymbol{t}, \boldsymbol{u} \in \mathbb{R}^2, \end{aligned}$$

where $g, g_1, g_2, g_{12} : \mathbb{R}^2 \to \mathbb{R}$ are measurable functions.

Then X has stationary rectangular increments: for all $\boldsymbol{t}_0 \in \mathbb{R}^2$,

$$\left\{\Delta_{t_1}^{(1)} \Delta_{t_2}^{(2)} X(\boldsymbol{t}_0)\right\}_{\boldsymbol{t} \in \mathbb{R}_+^2} \overset{\text{fdd}}{=} \left\{\Delta_{t_1}^{(1)} \Delta_{t_2}^{(2)} X(\boldsymbol{0})\right\}_{\boldsymbol{t} \in \mathbb{R}_+^2}.$$

If $g_1 = g_2 = 0$, then X also has stationary ordinary increments.

Let $0 < \alpha \le 2$ be the same. Assume

$(\mathsf{G})_\alpha^0$ The functions $g(\boldsymbol{u})$ and $g_0(\boldsymbol{u}) = \rho(\boldsymbol{u})^\chi \ell(\boldsymbol{u})$, where

$$\rho(\boldsymbol{u}) = |u_1|^{q_1} + |u_2|^{q_2}, \qquad \ell(\boldsymbol{u}) = \ell(\lambda^{\frac{1}{q_1}} u_1, \lambda^{\frac{1}{q_2}} u_2) \text{ for all } \lambda > 0,\ \boldsymbol{u} \in \mathbb{R}_0^2,$$

for the parameters $\chi \in \mathbb{R}$, $q_1, q_2 > 0$ with $Q = \frac{1}{q_1} + \frac{1}{q_2}$ satisfying

$$-\frac{1}{\alpha} Q < \chi < \left(1 - \frac{1}{\alpha}\right) Q, \qquad \chi \neq 0, \tag{3}$$

have continuous partial derivatives up to the 2nd order at all $\boldsymbol{u} \in \mathbb{R}_0^2$. Moreover, as $\boldsymbol{u} \to \boldsymbol{0}$,

$$g(\boldsymbol{u}) = g_0(\boldsymbol{u}) + o(\rho(\boldsymbol{u})^\chi),$$

$$\partial_{u_i} g(\boldsymbol{u}) = \partial_{u_i} g_0(\boldsymbol{u}) + o(\rho(\boldsymbol{u})^{\chi - \frac{1}{q_i}}),\ i = 1, 2,$$

$$\partial_{u_1} \partial_{u_2} g(\boldsymbol{u}) = \partial_{u_1} \partial_{u_2} g_0(\boldsymbol{u}) + o(\rho(\boldsymbol{u})^{\chi - Q})$$

and for all $\boldsymbol{u} \in \mathbb{R}_0^2$,

$$|g_0(\boldsymbol{u})| \le C \rho(\boldsymbol{u})^\chi,$$

$$|\partial_{u_i} g_0(\boldsymbol{u})| \le C \rho(\boldsymbol{u})^{\chi - \frac{1}{q_i}},\ i = 1, 2,$$

$$|\partial_{u_1} \partial_{u_2} g_0(\boldsymbol{u})| \le C \rho(\boldsymbol{u})^{\chi - Q}.$$

$(\mathrm{G})^0_\alpha$ (cont)

If $\alpha \ge 1$, then for all $\delta > 0$,

$$\int_{\|\boldsymbol{u}\|>\delta} \Big(\sum_{i=1}^{2} |\partial_{u_i} g(\boldsymbol{u})|^\alpha + |\partial_{u_1}\partial_{u_2} g(\boldsymbol{u})|^\alpha \Big) \mathrm{d}\boldsymbol{u} < \infty. \tag{4}$$

If $\alpha < 1$, then there exist $\delta_0 > 0$ and functions $\bar{g}_i(\boldsymbol{u})$, $\bar{g}_{12}(\boldsymbol{u})$, $\boldsymbol{u} \in \mathbb{R}^2_+$, monotone decreasing in each u_j, $j = 1, 2$, and satisfying

$$|\partial_{u_i} g(\boldsymbol{u})| \le \bar{g}_i(|u_1|, |u_2|), \quad |\partial_{u_1}\partial_{u_2} g(\boldsymbol{u})| \le \bar{g}_{12}(|u_1|, |u_2|), \quad \|\boldsymbol{u}\| > \delta_0,$$

so that (4) holds with $\partial_{u_i} g$, $\partial_{u_1}\partial_{u_2} g$ replaced by $\bar{g}_i$, $\bar{g}_{12}$, $i = 1, 2$.

Example 1[12]

A moving average fractional α-stable RF for $H \in (0,1)$, $0 < \alpha \leq 2$, is

$$X(\boldsymbol{t}) = \int_{\mathbb{R}^2} \left\{ \|\boldsymbol{t} - \boldsymbol{u}\|^{H - \frac{2}{\alpha}} - \|\boldsymbol{u}\|^{H - \frac{2}{\alpha}} \right\} W_\alpha(\mathrm{d}\boldsymbol{u}), \quad \boldsymbol{t} \in \mathbb{R}^2.$$

If $\alpha = 2$ then it is called a fractional Brownian RF and

$$\mathbb{E} X(\boldsymbol{t}) X(\boldsymbol{s}) = \mathbb{E}|X(\boldsymbol{e}_1)|^2 \frac{1}{2} \left(\|\boldsymbol{t}\|^{2H} + \|\boldsymbol{s}\|^{2H} - \|\boldsymbol{t} - \boldsymbol{s}\|^{2H} \right), \quad \boldsymbol{t}, \boldsymbol{s} \in \mathbb{R}^2,$$

where $\mathbb{E}|X(\boldsymbol{e}_1)|^2 < \infty$.

[12] Takenaka, S. (1991). Integral-geometric construction of self-similar stable processes. Nagoya Math. J. 123, 1–12.

Example 2

Isotropic fractional Laplace or Matérn RF

Example 3[13]

Anisotropic fractional heat operator RF
(with $g_0(\boldsymbol{u}) = (|u_1| + |u_2|^2)^\chi \ell(\boldsymbol{u})$, $\boldsymbol{u} \in \mathbb{R}_0^2$, $-\frac{3}{4} < \chi < \frac{3}{4}$, $\chi \neq 0$)

[13] Kelbert, M.Ya., Leonenko, N.N., Ruiz-Medina, M.D. (2005). Fractional random fields associated with stochastic fractional heat equations. Adv. Appl. Probab. 37, 108–133.

3. Main results: γ-rectangent limits

If $p_i = q_i(Q-\chi) > 0$, $i = 1,2$, with $P = \frac{1}{p_1} + \frac{1}{p_2}$, then $\frac{\alpha}{1+\alpha} < P < \alpha$, $P \neq 1$ $\iff$ (3) in $(\mathrm{G})^0_\alpha$.

Summary

Assume X satisfies $(\mathrm{M})_\alpha$, $(\mathrm{G})_\alpha$, $(\mathrm{G})^0_\alpha$, $P \neq \alpha - \frac{\alpha}{p_i}$, $i = 1,2$, $0 < \alpha \leq 2$. Then

- R_γ, $\gamma > 0$, exist, undergo a transition at $\gamma_0 := \frac{p_1}{p_2} = \frac{q_1}{q_2}$, and, are
- α-stable RFs with stationary rectangular increments,
- operator scaling RFs[14] with $H_\gamma > 0$ such that $C_{\lambda,\gamma} = \lambda^{H_\gamma}$ in (2):

$$\{R_\gamma(\lambda t_1, \lambda^\gamma t_2)\}_{\boldsymbol{t}\in\mathbb{R}^2_+} \overset{\mathrm{fdd}}{=} \{\lambda^{H_\gamma} R_\gamma(\boldsymbol{t})\} \text{ for all } \lambda > 0,$$

- multi self-similar RFs[15] with index $\boldsymbol{H}$, where one of $H_i \geq 0$, $i = 1,2$, equals to either 1 or 0, if $\gamma \neq \gamma_0$:

$$\{R_\gamma(\lambda_1 t_1, \lambda_2 t_2)\}_{\boldsymbol{t}\in\mathbb{R}^2_+} \overset{\mathrm{fdd}}{=} \{\lambda_1^{H_1}\lambda_2^{H_2} R_\gamma(\boldsymbol{t})\}_{\boldsymbol{t}\in\mathbb{R}^2_+} \text{ for all } \lambda_1, \lambda_2 > 0.$$

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[14] Biermé, H., Meerschaert, M.M, Scheffler, H.-P. (2007). Operator scaling stable random fields. Stochastic Process. Appl. 117, 312–332.

[15] Genton, M.G., Perrin, O., Taqqu, M.S. (2007). Self-similarity and Lamperti transformation for random fields. Stoch. Models 23, 397–411.

4. Extensions

(i) What are γ-rectangent RFs of X with 'smooth' g at $\mathbf{0}$?

(ii) When do γ-rectangent RFs of X agree with α-stable Lévy sheet?

(iii) What are γ-tangent RFs of X?

(iv) Functional convergence instead of $\xrightarrow{\text{fdd}}$.

(v) Extension to Lévy driven fractional RFs on $\mathbb{R}^d$, $d \geq 3$.

(vi) Application to statistical estimation of H_1, H_2 using for some γ, r:

$$\sum_{(\frac{i_1}{n}, \frac{i_2}{n^\gamma}) \in [0,1)^2} \Big|\Delta^{(1)}_{\frac{1}{n}} \Delta^{(2)}_{\frac{1}{n^\gamma}} X\Big(\frac{i_1}{n}, \frac{i_2}{n^\gamma}\Big)\Big|^r.$$

Thank you for your attention

Example 2

For $\boldsymbol{t} \in \mathbb{R}^2$, let

$$X(\boldsymbol{t}) = \begin{cases} Y(\boldsymbol{t}), & \chi < 0, \\ Y(\boldsymbol{t}) - Y(\boldsymbol{0}), & \chi > 0, \end{cases}$$

where

$$Y(\boldsymbol{t}) = \frac{2^{1+\chi}}{c^\chi \Gamma(-\chi)} \int_{\mathbb{R}^2} \|\boldsymbol{t}-\boldsymbol{u}\|^\chi K_\chi(c\|\boldsymbol{t}-\boldsymbol{u}\|) M(\mathrm{d}\boldsymbol{u}),$$

K_χ denotes a modified Bessel function of the 2nd kind of order χ and $-\frac{1}{\alpha} < \chi < 1 - \frac{1}{\alpha}$, $\chi \neq 0$, $c > 0$. Then $(\mathrm{G})^0_\alpha$ holds with $g_0(\boldsymbol{u}) = \|\boldsymbol{u}\|^{2\chi}$, $\boldsymbol{u} \in \mathbb{R}^2_0$.

- If $\alpha = 2$, then Y has Matérn covariance

$$\mathbb{E}Y(\boldsymbol{0})Y(\boldsymbol{t}) = \mathbb{E}|Y(\boldsymbol{0})|^2 \frac{(c\|\boldsymbol{t}\|)^{1+2\chi} K_{1+2\chi}(c\|\boldsymbol{t}\|)}{\Gamma(1+2\chi)2^{2\chi}}, \quad \boldsymbol{t} \in \mathbb{R}^2,$$

  where $\mathbb{E}|Y(\boldsymbol{0})|^2 < \infty$. Moreover, Y is a stationary solution of

$$(c^2 - \Delta)^{1+\chi} Y(\boldsymbol{t}) = \dot{M}(\boldsymbol{t}), \quad \boldsymbol{t} \in \mathbb{R}^2,$$

  with $\Delta := \partial^2_{t_1} + \partial^2_{t_2}$ and Lévy white noise $\dot{M}$[16,17].

[16] Whittle, P. (1963). Stochastic processes in several dimensions. Bull. Int. Statist. Inst. 40, 974–997.

[17] Bolin, D. (2014). Spatial Matérn fields driven by non-Gaussian noise. Scand. J. Statist. 41, 557–579.

Example 3[18]

A stationary solution of

$$(c_1 + \Delta_{12})^{\chi+\frac{3}{2}} X(\boldsymbol{t}) = \dot{M}(\boldsymbol{t}), \quad \boldsymbol{t} \in \mathbb{R}^2,$$

with $\Delta_{12} := \partial_{t_1} - c_2^2 \partial_{t_2}^2$ and Gaussian white noise $\dot{M}$ is given by

$$X(\boldsymbol{t}) = \int_{\mathbb{R}^2} g(\boldsymbol{t} - \boldsymbol{u}) M(\mathrm{d}\boldsymbol{u}), \quad \boldsymbol{t} \in \mathbb{R}^2,$$

where $g(\boldsymbol{u}) = \mathrm{e}^{-c_1 u_1} g_0(\boldsymbol{u})$ with $g_0(\boldsymbol{u}) = \rho(\boldsymbol{u})^\chi \ell(\boldsymbol{u})$ and

$$\rho(\boldsymbol{u}) = |u_1| + |u_2|^2, \quad \ell(\boldsymbol{u}) = \frac{\left(\frac{u_1}{\rho(\boldsymbol{u})}\right)^\chi \mathrm{e}^{-\frac{1}{4c_2^2}\left(\frac{\rho(\boldsymbol{u})}{u_1} - 1\right)}}{2^{\frac{1}{2}} (2\pi)^{\frac{3}{2}} c_2 \Gamma(\chi + \frac{3}{2})} I(u_1 > 0), \quad \boldsymbol{u} \in \mathbb{R}^2,$$

satisfies $(\mathrm{G})_\alpha^0$ for $c_1, c_2 > 0$, $-\frac{3}{4} < \chi < \frac{3}{4}$, $\chi \neq 0$.

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[18] Kelbert, M.Ya., Leonenko, N.N., Ruiz-Medina, M.D. (2005). Fractional random fields associated with stochastic fractional heat equations. Adv. Appl. Probab. 37, 108–133.

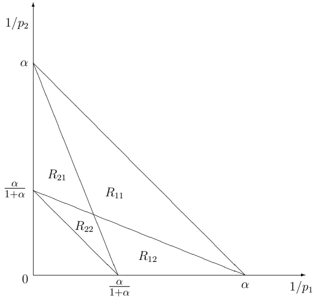
Proposition

Assume $(\mathsf{M})_\alpha$, $0 < \alpha \le 2$, and $f_\lambda \in L^\alpha(\mathbb{R}^2)$, $\lambda > 0$. If for some $\mu > 0$, the functions

$$\tilde{f}_\lambda(\boldsymbol{u}) := \lambda^{\frac{1}{\alpha}(1+\mu)} f_\lambda(\lambda u_1, \lambda^\mu u_2), \quad \boldsymbol{u} \in \mathbb{R}^2,$$

tend to the limit f_0 in $L^\alpha(\mathbb{R}^2)$ as $\lambda \downarrow 0$, then

$$\int_{\mathbb{R}^2} f_\lambda(\boldsymbol{u}) M(\mathrm{d}\boldsymbol{u}) \xrightarrow{\mathrm{d}} \int_{\mathbb{R}^2} f_0(\boldsymbol{u}) W_\alpha(\mathrm{d}\boldsymbol{u}) \quad \text{as } \lambda \downarrow 0.$$



Parameter region	Multi self-similarity index $\boldsymbol{H}$ of R_γ	
	$\gamma < \gamma_0$	$\gamma > \gamma_0$
R_{11}	$(1, H_2)$	$(\tilde{H}_1, 1)$
R_{12}	$(H_1, 0)$	$(\tilde{H}_1, 1)$
R_{21}	$(1, H_2)$	$(0, \tilde{H}_2)$
R_{22}	$(H_1, 0)$	$(0, \tilde{H}_2)$

Definition

A standard fractional Brownian sheet $\{B_{\boldsymbol{\mathcal{H}}}(\boldsymbol{t})\}_{\boldsymbol{t}\in\mathbb{R}_+^2}$ with $\boldsymbol{\mathcal{H}} = (\mathcal{H}_1, \mathcal{H}_2) \in [0,1]^2$ is a Gaussian random field that has mean zero and

$$\mathbb{E}B_{\boldsymbol{\mathcal{H}}}(\boldsymbol{t})B_{\boldsymbol{\mathcal{H}}}(\boldsymbol{s}) = \prod_{i=1}^{2} C_{\mathcal{H}_i}(t_i, s_i), \quad \boldsymbol{t}, \boldsymbol{s} \in \mathbb{R}_+^2,$$

where for all $t, s \in \mathbb{R}_+$,

$$C_{\mathcal{H}}(t,s) = \frac{1}{2}(t^{2\mathcal{H}} + s^{2\mathcal{H}} - |t-s|^{2\mathcal{H}}) \text{ if } \mathcal{H} \in (0,1],$$
$$C_0(t,s) = \lim_{\mathcal{H}\downarrow 0} C_{\mathcal{H}}(t,s) = 1 - \frac{1}{2}I(t \neq s).$$

- Extension to $\mathcal{H}_1 \wedge \mathcal{H}_2 = 0$[19].
- $B_{\mathcal{H}_1} = \{B_{\boldsymbol{\mathcal{H}}}(t,1)\}_{t\in\mathbb{R}_+}$ is a standard fractional Brownian motion with Hurst parameter $\mathcal{H}_1 \in [0,1]$.
- $B_0 \overset{\text{fdd}}{=} \{2^{-\frac{1}{2}}(W(t) - W(0))\}_{t\in\mathbb{R}_+}$, where $W(t)$, $t \geq 0$, are independent $N(0,1)$-distributed random variables.

[19] Surgailis, D. (2020). Stochastic Process. Appl. 130, 7518–7546.

(i) What are γ-rectangent RFs of X with 'smooth' g at $\mathbf{0}$?

Proposition

Let $0 < \alpha \le 2$. Assume X satisfies (M)$_\alpha$ with (i), (ii) replaced by

(i') $\sigma = 0$, $\sup_{y>0} y^\alpha \nu(\{u \in \mathbb{R} : |u| > y\}) < \infty$ if $\alpha < 2$,

(ii') $\sigma > 0$, $\int_{\mathbb{R}} y^2 \nu(\mathrm{d}y) < \infty$ if $\alpha = 2$,

and (G)$_\alpha$ and

$$\lim_{t_1 \vee t_2 \downarrow 0} \int_{\mathbb{R}^2} \left| (t_1 t_2)^{-1} \Delta_{t_1}^{(1)} \Delta_{t_2}^{(2)} g(\boldsymbol{u}) - \partial_{u_1} \partial_{u_2} g(\boldsymbol{u}) \right|^\alpha \mathrm{d}\boldsymbol{u} = 0. \tag{5}$$

Then at any $\boldsymbol{t}_0 \in \mathbb{R}^2$, for any $\gamma > 0$, as $\lambda \downarrow 0$,

$$\left\{ \lambda^{-1-\gamma} \Delta_{\lambda t_1}^{(1)} \Delta_{\lambda^\gamma t_2}^{(2)} X(\boldsymbol{t}_0) \right\}_{\boldsymbol{t} \in \mathbb{R}_+^2} \overset{\text{fdd}}{\to} \left\{ t_1 t_2 \int_{\mathbb{R}^2} \partial_{u_1} \partial_{u_2} g(\boldsymbol{u}) M(\mathrm{d}\boldsymbol{u}) \right\}_{\boldsymbol{t} \in \mathbb{R}_+^2}.$$

- We can replace (5) by (G)$_\alpha^0$ with $P > \alpha$.

(ii) When γ-rectangent RFs of X agree with symmetric α-stable Lévy sheet?

Proposition

Let $0 < \alpha \le 2$. Assume X satisfies (M)$_\alpha$, (G)$_\alpha$ and

$$g[\mathbf{0}] = \sum_{i=1}^{4} \lim_{\boldsymbol{u}\to\mathbf{0},\, \boldsymbol{u}\in Q_i} g(\boldsymbol{u})\,\mathrm{sign}(u_1u_2) \neq 0,$$

where Q_i is the i-th quadrant, $i = 1, 2, 3, 4$. Moreover,

$$\int_{\mathbb{R}^2} \left|\Delta^{(1)}_{t_1}\Delta^{(2)}_{t_2} g(-\boldsymbol{u}) - g[\mathbf{0}] I(\boldsymbol{u} \in [\mathbf{0}, \boldsymbol{t}])\right|^{\alpha} \mathrm{d}\boldsymbol{u} = o(t_1t_2), \quad t_1 \vee t_2 \downarrow 0.$$

Then for any $\gamma > 0$, as $\lambda \downarrow 0$,

$$\left\{\lambda^{-\frac{1}{\alpha}(1+\gamma)}\Delta^{(1)}_{\lambda t_1}\Delta^{(2)}_{\lambda^\gamma t_2} X(\mathbf{0})\right\}_{\boldsymbol{t}\in\mathbb{R}^2_+} \overset{\text{fdd}}{\to} \left\{g[\mathbf{0}]W_\alpha([\mathbf{0},\boldsymbol{t}])\right\}_{\boldsymbol{t}\in\mathbb{R}^2_+}.$$

- For example, $g(\boldsymbol{u}) = \sum_{i,j=\pm 1} g_{ij} I(\boldsymbol{u} \in \mathbb{R}^2_{ij})$, $\boldsymbol{u} \in \mathbb{R}^2$.

(iii) What are γ-tangent RFs of X?

Theorem

Assume X satisfies (M)$_\alpha$, (G)$_\alpha$ with $g_i = 0$, $i = 1, 2$, $0 < \alpha \le 2$. Let

$$g(\boldsymbol{u}) = g_0(\boldsymbol{u})(1 + o(1)), \quad \boldsymbol{u} \to \boldsymbol{0},$$

where

$$g_0(\boldsymbol{u}) = \rho(\boldsymbol{u})^{\chi}\ell(\boldsymbol{u}) \text{ with } \rho(\boldsymbol{u}) = |u_1|^{q_1} + |u_2|^{q_2}$$

and $\ell(\boldsymbol{u}) = \ell(\lambda^{\frac{1}{q_1}} u_1, \lambda^{\frac{1}{q_2}} u_2)$, $\lambda > 0$, $\boldsymbol{u} \in \mathbb{R}^2_0$, for the parameters $\chi < 0$ and $q_i > 0$, $i = 1, 2$, with $Q = \frac{1}{q_1} + \frac{1}{q_2}$ satisfying

$$-\frac{1}{\alpha}Q < \chi < \frac{1}{q_1 \vee q_2} - \frac{1}{\alpha}Q. \tag{6}$$

Moreover, let g, g_0 have partial derivatives on $\mathbb{R}^2_0$ and $|g(\boldsymbol{u})| + |g_0(\boldsymbol{u})| \le C\rho(\boldsymbol{u})^{\chi}$, $|\partial_{u_i} g(\boldsymbol{u})| + |\partial_{u_i} g_0(\boldsymbol{u})| \le C\rho(\boldsymbol{u})^{\chi - \frac{1}{q_i}}$, $i = 1, 2$, for all $\boldsymbol{u} \in \mathbb{R}^2_0$.

Theorem (cnt)

Then at any $\boldsymbol{t}_0 \in \mathbb{R}^2$ *for any* $\gamma > 0$*, as* $\lambda \downarrow 0$*,*

$$\left\{\lambda^{-(1\wedge\frac{\gamma}{\gamma_0})(\chi+\frac{1}{\alpha}Q)q_1}\left(X(t_1^0+\lambda t_1, t_2^0+\lambda^\gamma t_2) - X(t_1^0, t_2^0)\right)\right\}_{\boldsymbol{t}\in\mathbb{R}_+^2} \overset{\mathsf{fdd}}{\to} T_\gamma,$$

where

$$T_\gamma = \begin{cases} T_+, & \gamma > \gamma_0, \\ T_0, & \gamma = \gamma_0, \\ T_-, & \gamma < \gamma_0, \end{cases}$$

with $\gamma_0 = \frac{q_1}{q_2}$ *and*

$$T_0(\boldsymbol{t}) = \int_{\mathbb{R}^2} \left\{g_0(\boldsymbol{t}-\boldsymbol{u}) - g_0(-\boldsymbol{u})\right\} W_\alpha(\mathrm{d}\boldsymbol{u}),$$

$$T_+(\boldsymbol{t}) = T_0(t_1, 0), \quad T_-(\boldsymbol{t}) = T_0(0, t_2), \quad \boldsymbol{t} \in \mathbb{R}_+^2.$$