Stability criteria for Intercating Particle Systems via Toom contours

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joint work with Jan M. Swart and Cristina Toninelli

August 30, 2022

Dauphine | PSL CEREMADE UMR CNRS 7534 cnrs

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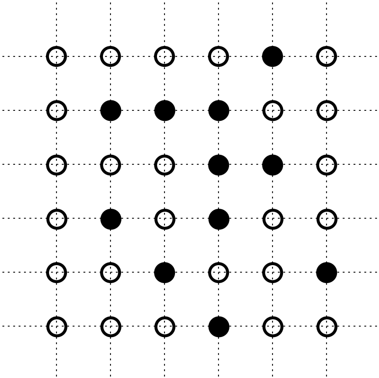
discrete-time Markov chain with state space $\{0,1\}^{\mathbb{Z}^d}$ characterized by a monotone map $\varphi : \{0,1\}^{\mathbb{Z}^d} \mapsto \{0,1\}$ that depends on finitely many coordinates



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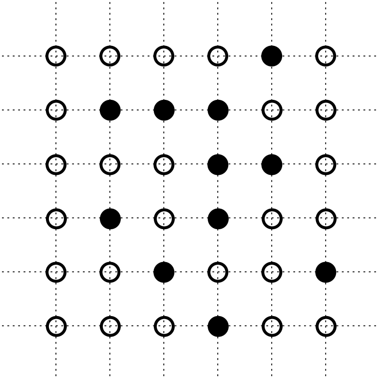

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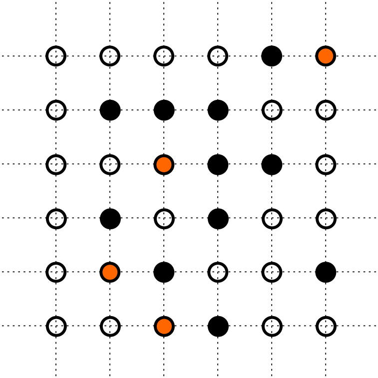

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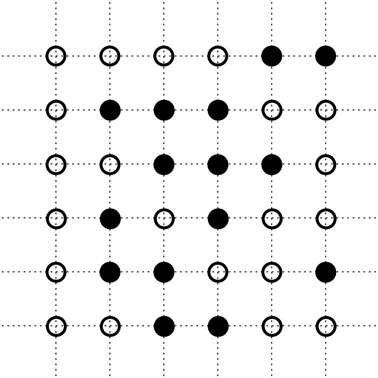

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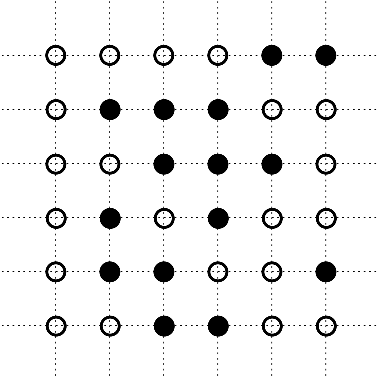

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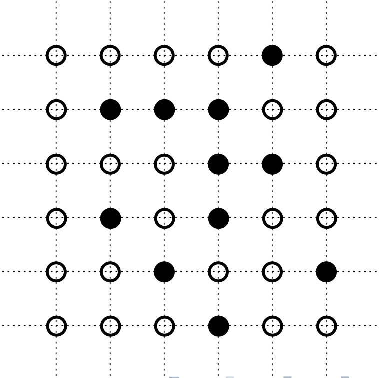

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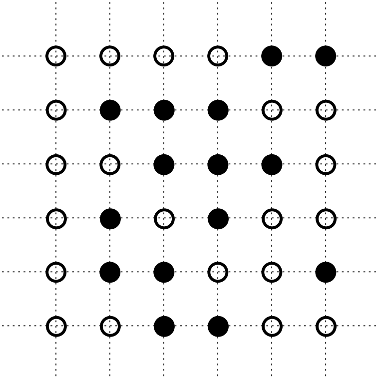

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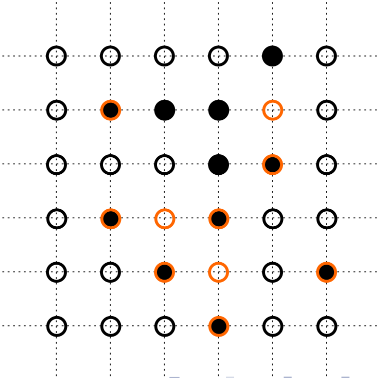

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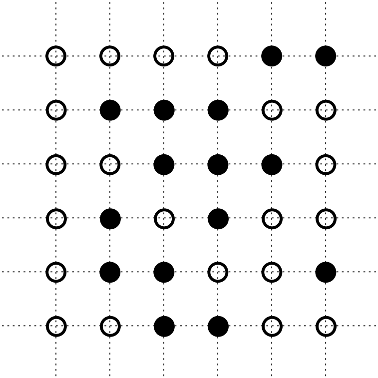

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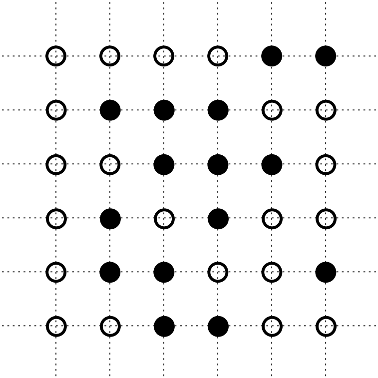

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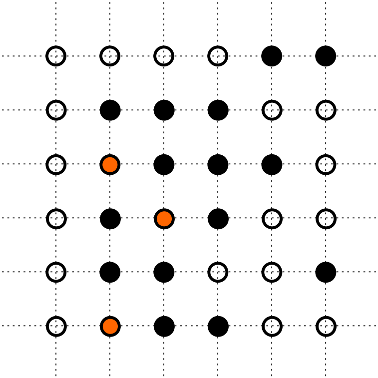

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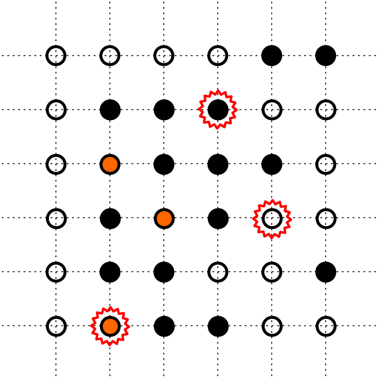

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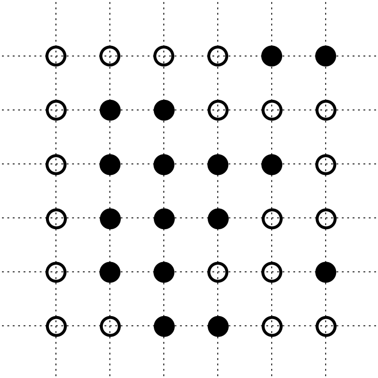

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Toom's stability theorem

$$\text{CA applies} \begin{cases} \varphi & \text{with prob. } 1-p \\ \varphi^0 & \text{with prob. } p \end{cases} \text{(at each space-time point)}$$

Monotone CA **stable** against random perturbation:

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Theorem (Toom, 1980)

The Monotone CA is ***stable*** *against random perturbations if and only if its monotone map is an* ***eroder****.*

Eroder: in the absence of defective sites a finite set of zeros disappear in finite steps

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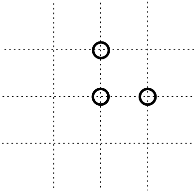

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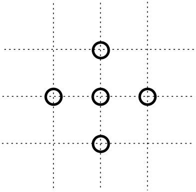

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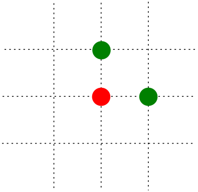

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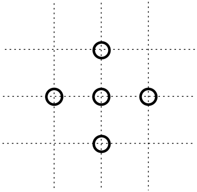

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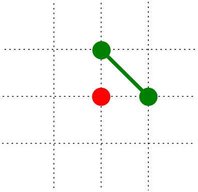

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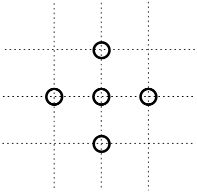

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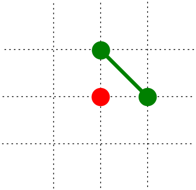

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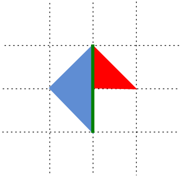

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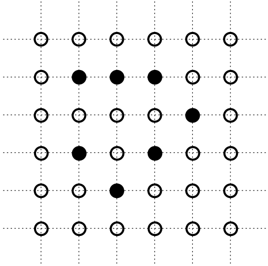

two exponential clocks at each vertex:

- rate λ (birth)
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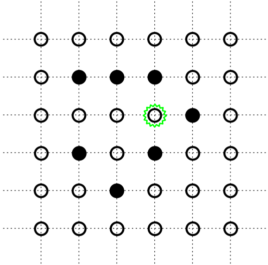

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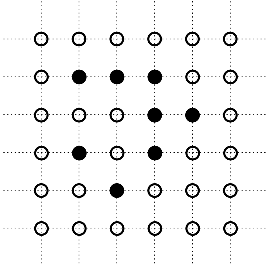

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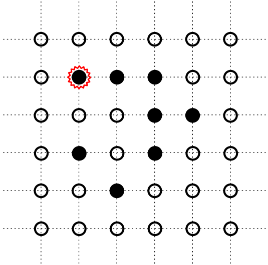

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Gray (99): sufficient conditions for stability of IPS and examples for

- φ eroder, but IPS unstable
- φ non-erdoer, but IPS stable

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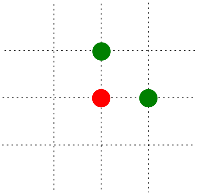

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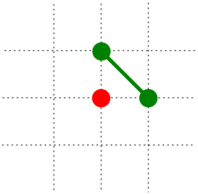

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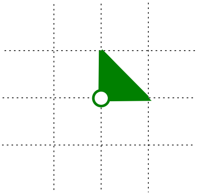

Bounds on critical parameter

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Our bound:

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