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Théorie des ensembles aléatoires: conditionnement et applications en optimisation et en finance.

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The problem of hedging (or super-hedging) a European claim

In discrete time $t = 0, \cdots, T$, let $(\Omega, (\mathcal{F}_t)_{t=0,\cdots,T}, P)$ be a discrete-time complete stochastic basis. Consider a $\mathcal{F}_T$-measurable random variable ξ_T we interpret as the payoff of some European option, i.e. a financial contract delivering the wealth ξ_T at time T.

The general problem is to solve the following : find a self-financing portfolio process $(V_t)_{t=0,\cdots,T}$ such that $V_T \geq \xi_T$ (or $V_T = \xi_T$ for an exact replication). We say that the initial value V_0 is a super-hedging price.

We are interested in the infimum of the super-hedging prices.

Portfolio processes and self-financing condition

Suppose that the financial market is composed of one bond of (discounted) price $S^1 = 1$ and $d-1 \geq 1$ risky assets of prices $(S^i)_{i=2,\cdots,d}$.

A financial strategy $\theta \in \mathbf{R}^d$ is a stochastic process where θ_t^i is the number of assets number $i = 1, \cdots, d$ held by a portfolio manager.

Portfolio processes and self-financing condition : without transaction costs

The liquidation value of the financial strategy $\theta \in \mathbf{R}^d$ at time t is given by

$$L_t = L_t^\theta = \theta_t S_t = \sum_{i=1}^{d} \theta_t^i S_t^i.$$

The portfolio-process is said self-financing if, for all $t = 1, \cdots, T$,

$$\theta_{t-1} S_t = \theta_t S_t$$

or equivalently $\Delta L_t = L_t - L_{t-1}$ satisfies :

$$\Delta L_t = \theta_{t-1} \Delta S_t, t = 1, \cdots, T.$$

FTAP without transaction costs

Without transaction costs, the infimum super-replicating price of an European option payoff ξ_T is characterized under a no-arbitrage condition NA.

Definition

An arbitrage opportunity is a self-financing portfolio process $V_t = L_t^\theta$, $t = 1, \cdots, T$, such that $V_0 = 0$, $V_T \geq 0$ a.s. and $P(V_T > 0) > 0$.

Definition

NA : there is no arbitrage opportunity.

FTAP without transaction costs

Theorem (Dalang–Morton–Willinger)

In discrete time $t = 1, \cdots, T$, with $S^1 = 1$, NA holds if and only if there exists $Q \sim P$ such that the (discounded) asset price $(S_t)_{t=1,\cdots,T}$ is a Q-martingale.

Theorem

With $S^1 = 1$, let $\mathcal{M}(P)$ be the set of all risk-neutral probability measures for S and let ξ_T be an European option payoff. Then, the infimum super-replicating price of ξ_T is given by

$$V_0^* = \sup_{Q \in \mathcal{M}(P)} E_Q(\xi_T).$$

Portfolio processes and self-financing condition : with transaction costs

The general framework derived from the Kabanov model is the following.
A portfolio process is expressed in physical units, i.e. $V_t = \theta_t$ and the liquidation value $L_t = L_t^{V_t}$ is not always simple to express.
We consider the associated set-valued stochastic process $(G_t)_{t=0,\cdots,T}$ in $\mathbf{R}^d$ defined as

$$G_t := \{x \in \mathbf{R}^d : \; L_t(x) \geq 0\}.$$

If $\omega \mapsto L_t(\omega, x)$ is $\mathcal{F}_t$-measurable and $x \mapsto L_t(\omega, x)$ is upper semi-continuous, we may show that $G_t(\omega)$ is closed a.s.(ω) and measurable, where the measurability is understood in the graph sense :

$$\mathrm{graph}(G_t) := \{(\omega, x) : \; x \in G_t(\omega)\} \in \mathcal{F}_t \times \mathcal{B}(\mathbf{R}^d), \; t = 1, \cdots, T.$$

Portfolio processes and self-financing condition : with transaction costs

Moreover, with $e_1 = (1, 0 \cdots, 0) \in \mathbf{R}^d$, we have

$$L_t(z) := \sup \{\alpha \in \mathbf{R} : z - \alpha e_1 \in G_t\} = \max \{\alpha \in \mathbf{R} : z - \alpha e_1 \in G_t\},$$

i.e. $L_t(z)$ is the maximum amount of cash α we may obtain when we change $z = (z - \alpha e_1) + \alpha e_1$ into αe_1.

Similarly, if we define $C_t(z) = -L_t(-z)$, we obtain that

$$C_t(z) = \inf\{\alpha \in \mathbf{R} : \ \alpha e_1 - z \in G_t\} = \min\{\alpha \in \mathbf{R} : \ \alpha e_1 - z \in G_t\},$$

i.e. $C_t(z)$ is the minimum cost α expressed in cash we need to buy the financial position $z \in \mathbf{R}^d$. Indeed, we write $\alpha e_1 = z + (\alpha e_1 - z)$.

Portfolio processes and self-financing condition : with transaction costs

Naturally, $C_t(z) = C_t(S_t, z)$ depends on the available quantities and prices for the risky assets, described by an exogenous vector-valued $\mathcal{F}_t$-measurable random variable S_t of $\mathbf{R}_+^m$, $m \geq d$, and on the quantities $z \in \mathbf{R}^d$ to be traded.

We generally suppose that $m \geq d$ as an asset may be described by several prices and quantities offered by the market, e.g. bid and ask prices, or several pair of bid and ask prices of an order book and the associated quantities offered by the market.

Portfolio processes and self-financing condition : with transaction costs

The self-financing condition is :

$$\Delta V_t \in -G_t,\ t = 0, \cdots, T$$

i.e. $V_{t-1} = V_t + (-\Delta V_t)$ is such that $L_t(-\Delta V_t) \geq 0$ so that we may cancel the position $-\Delta V_t$ and change V_{t-1} into V_t for free.

The super-hedging problem with proportional transaction costs

Some no-arbitrage conditions are introduced for physical self-financing portfolio processes in the spirit of NA, e.g. the robust NA^r condition, see the Kabanov model, and we have :

Theorem

NA^r holds if and only if there exists strictly consistent price systems (SCPS), i.e. martingales Z of $\mathbf{R}^d$ such that, for all $t = 0, \cdots, d$, $Z_t \in G_t^* = \{y \in \mathbf{R}^d : \ xy \geq 0, \forall x \in G_t\}$.

Then, there exists a minimal price in cash for the European claim $\xi_T \in \mathbf{R}^d$ given by

$$\sup_{Z \in SCPS,\, Z_0 e_1 = 1} EZ_T \xi_T.$$

The super-hedging problem for general transaction costs

In practice the transaction costs are not necessary linear, see the case of order books or fixed costs.

The model is not linear so that we cannot expect dual elements characterizing a no-arbitrage condition that allow us to dually characterize the super-hedging prices.

Moreover, the no-arbitrage conditions we can imagine seem to be rather artificial, see the case of the Kabanov model where several distinct no-arbitrage exist and are difficult to compare.

A new approach based on random set

A random set is a set-valued process $(G_t)_{t=0,\cdots,T}$ such that G_t is a mapping defined on Ω with values $G_t(\omega)$ which are subsets of $\mathbf{R}^d$, $d \geq 1$.

Definition

We say that the set-valued process $(G_t)_{t=0,\cdots,T}$ is graph-measurable with respect to the filtration $(\mathcal{F}_t)_{t=0,\cdots,T}$ is, for all $t \leq T$, we have :

$$\mathrm{graph}(G_t) := \{(\omega, x) : \; x \in G_t(\omega)\} \in \mathcal{F}_t \times \mathcal{B}(\mathbf{R}^d).$$

Conditioning the random sets : conditional core

In mathematical finance, we meet some problems such as :

Find the set of all $\mathcal{F}_{t-1}$-measurable random variables V_{t-1} such that $V_{t-1} \in \Gamma_t$ a.s. where Γ_t is an $\mathcal{F}_t$-graph-measurable random set. We denote this family by $L^0(\Gamma_t, \mathcal{F}_{t-1})$.

Example 1 : When $d = 1$, if $\Gamma_t(\omega) = [V_t(\omega), \infty)$ where V_t is $\mathcal{F}_t$-measurable, then $V_{t-1} \in L^0(\Gamma_t, \mathcal{F}_{t-1})$ iff $V_{t-1} \geq V_t$ a.s..
We get that $L^0(\Gamma_t, \mathcal{F}_{t-1}) = L^0(\mathbf{m}(\Gamma_t | \mathcal{F}_{t-1}), \mathcal{F}_{t-1})$ where

$$\mathbf{m}(\Gamma_t | \mathcal{F}_{t-1}) = [\operatorname{ess\,sup}_{\mathcal{F}_{t-1}}(V_t), \infty).$$

Example 2 : When $d = 1$, if $\Gamma_t(\omega) = (-\infty, V_t(\omega)])$ where V_t is $\mathcal{F}_t$-measurable, then $V_{t-1} \in L^0(\Gamma_t, \mathcal{F}_{t-1})$ iff $V_{t-1} \leq V_t$ a.s..
We get that $L^0(\Gamma_t, \mathcal{F}_{t-1}) = L^0(\mathbf{m}(\Gamma_t | \mathcal{F}_{t-1}), \mathcal{F}_{t-1})$ where

$$\mathbf{m}(\Gamma_t | \mathcal{F}_{t-1}) = (-\infty, \operatorname{ess\,inf}_{\mathcal{F}_{t-1}}(V_t)].$$

Conditioning the random sets : conditional core

Theorem

Suppose that Γ_t is a $\mathcal{F}_t$-graph-measurable random set which is a.s. closed. Then, there exists a largest $\mathcal{F}_{t-1}$-graph-measurable random set, denoted by $\mathbf{m}(\Gamma_t|\mathcal{F}_{t-1})$ and called $\mathcal{F}_{t-1}$-conditional core of Γ_t, such that $\mathbf{m}(\Gamma_t|\mathcal{F}_{t-1}) \subseteq \Gamma_t$ a.s.. Moreover, we have $L^0(\Gamma_t, \mathcal{F}_{t-1}) = L^0(\mathbf{m}(\Gamma_t|\mathcal{F}_{t-1}), \mathcal{F}_{t-1})$.

Conditioning the random sets : conditional closure and interior

The conditional closure concept arises from the following problem :

Let $h_{t-1}(\omega, x)$ be a random function such that $\omega \in \Omega \mapsto h_{t-1}(\omega, x)$ is $\mathcal{F}_{t-1}$-measurable and $x \in \mathbf{R}^d \mapsto h_{t-1}(\omega, x) \in \mathbf{R}$ is l.s.c. a.s.(ω). With a complete σ-algebra, we may say that h_{t-1} is a normal integrand. Then, compute

$$\mathrm{ess\,sup}_{\mathcal{F}_{t-1}}\{h_{t-1}(\omega, S_t(\omega)) : \; S_t \in L^0(\Gamma_t, \mathcal{F}_t)\}$$

where Γ_t is a $\mathcal{F}_t$-graph-measurable set-valued mapping .

Conditioning the random sets : conditional closure and interior

Theorem

Let Γ_t *be a* $\mathcal{F}_t$*-graph-measurable set-valued mapping. There exists a largest* $\mathcal{F}_{t-1}$*-graph-measurable set-valued mapping denoted by* $\mathrm{O}(\Gamma_t|\mathcal{F}_{t-1})$ *and called conditional interior, such that* $\mathrm{O}(\Gamma_t|\mathcal{F}_{t-1})$ *is a.s. open and* $\mathrm{O}(\Gamma_t|\mathcal{F}_{t-1}) \subseteq \Gamma_t$ *a.s..*

Corollary

Let Γ_t *be a* $\mathcal{F}_t$*-graph-measurable set-valued mapping. There exists a smallest* $\mathcal{F}_{t-1}$*-graph-measurable set-valued mapping denoted by* $\mathrm{cl}(\Gamma_t|\mathcal{F}_{t-1})$ *and called conditional closure, such that* $\mathrm{cl}(\Gamma_t|\mathcal{F}_{t-1})$ *is a.s. closed and* $\Gamma_t \subseteq \mathrm{cl}(\Gamma_t|\mathcal{F}_{t-1})$ *a.s..*

Application in random optimization

Theorem

Suppose that the random function $h_{t-1}(\omega, x)$ is a $\mathcal{F}_{t-1}$-normal integrand. Let Γ_t be a $\mathcal{F}_t$-graph-measurable set-valued mapping. Then,

$$\operatorname{ess\,sup}_{\mathcal{F}_{t-1}}\{h_{t-1}(\omega, S_t(\omega)) : \; S_t \in L^0(\Gamma_t, \mathcal{F}_t)\} = \sup_{z \in \mathrm{cl}(\Gamma_t | \mathcal{F}_{t-1})} h_{t-1}(\omega, z).$$

Application in random optimization

Theorem

Suppose that the random function $h_{t-1}(\omega, x)$ is a $\mathcal{F}_{t-1}$-normal integrand. Let S_t be a $\mathcal{F}_t$-measurable random variable. Then,

$$\operatorname{ess\,sup}_{\mathcal{F}_{t-1}} h_{t-1}(\omega, S_t(\omega)) = \sup_{z \in \mathrm{supp}(S_t|\mathcal{F}_{t-1})} h_{t-1}(\omega, z).$$

where $\mathrm{supp}(S_t|\mathcal{F}_{t-1})$ is the conditional support of S_t knowing $\mathcal{F}_{t-1}$.

Thank you for your attention !

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