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Théorie des ensembles aléatoires: conditionnement et applications en optimisation et en finance.

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The problem of hedging (or super-hedging) a European claim

In discrete time \( t = 0, \cdots, T \), let \((\Omega, (\mathcal{F}_t)_{t=0,\cdots, T}, P)\) be a discrete-time complete stochastic basis. Consider a \( \mathcal{F}_T \)-measurable random variable \( \xi_T \) we interpret as the payoff of some European option, i.e. a financial contract delivering the wealth \( \xi_T \) at time \( T \).

The general problem is to solve the following: find a self-financing portfolio process \((V_t)_{t=0,\cdots, T}\) such that \( V_T \geq \xi_T \) (or \( V_T = \xi_T \) for an exact replication). We say that the initial value \( V_0 \) is a super-hedging price.

We are interested in the infimum of the super-hedging prices.
Suppose that the financial market is composed of one bond of (discounted) price $S^1 = 1$ and $d - 1 \geq 1$ risky assets of prices $(S^i)_{i=2,\ldots,d}$.

A financial strategy $\theta \in \mathbb{R}^d$ is a stochastic process where $\theta^i_t$ is the number of assets number $i = 1, \ldots, d$ held by a portfolio manager.
Portfolio processes and self-financing condition: without transaction costs

The liquidation value of the financial strategy $\theta \in \mathbb{R}^d$ at time $t$ is given by

$$L_t = L_\theta = \theta_t S_t = \sum_{i=1}^{d} \theta^i_t S^i_t.$$ 

The portfolio-process is said self-financing if, for all $t = 1, \cdots, T$,

$$\theta_{t-1} S_t = \theta_t S_t$$

or equivalently $\Delta L_t = L_t - L_{t-1}$ satisfies:

$$\Delta L_t = \theta_{t-1} \Delta S_t, \ t = 1, \cdots, T.$$
Without transaction costs, the infimum super-replicating price of an European option payoff $\xi_T$ is characterized under a no-arbitrage condition NA.

**Definition**

An arbitrage opportunity is a self-financing portfolio process $V_t = L_t^\theta$, $t = 1, \cdots, T$, such that $V_0 = 0$, $V_T \geq 0$ a.s. and $P(V_T > 0) > 0$.

**Definition**

NA : there is no arbitrage opportunity.
FTAP without transaction costs

Theorem (Dalang–Morton–Willinger)

In discrete time $t = 1, \cdots , T$, with $S^1 = 1$, NA holds if and only if there exists $Q \sim P$ such that the (discounted) asset price $(S_t)_{t=1,\cdots,T}$ is a $Q$-martingale.

Theorem

With $S^1 = 1$, let $\mathcal{M}(P)$ be the set of all risk-neutral probability measures for $S$ and let $\xi_T$ be an European option payoff. Then, the infimum super-replicating price of $\xi_T$ is given by

$$V_0^* = \sup_{Q \in \mathcal{M}(P)} E_Q(\xi_T).$$
Portfolio processes and self-financing condition: with transaction costs

The general framework derived from the Kabanov model is the following.

A portfolio process is expressed in physical units, i.e. \( V_t = \theta_t \) and the liquidation value \( L_t = L^V_t \) is not always simple to express.

We consider the associated set-valued stochastic process \((G_t)_{t=0,\cdots,T}\) in \( \mathbb{R}^d \) defined as

\[
G_t := \{ x \in \mathbb{R}^d : L_t(x) \geq 0 \}.
\]

If \( \omega \mapsto L_t(\omega, x) \) is \( \mathcal{F}_t \)-measurable and \( x \mapsto L_t(\omega, x) \) is upper semi-continuous, we may show that \( G_t(\omega) \) is closed a.s.\((\omega)\) and measurable, where the measurability is understood in the graph sense:

\[
\text{graph}(G_t) := \{(\omega, x) : x \in G_t(\omega)\} \in \mathcal{F}_t \times \mathcal{B}(\mathbb{R}^d), \ t = 1, \cdots, T.
\]
Moreover, with $e_1 = (1, 0 \cdots, 0) \in \mathbb{R}^d$, we have

$$L_t(z) := \sup \{ \alpha \in \mathbb{R} : z - \alpha e_1 \in G_t \} = \max \{ \alpha \in \mathbb{R} : z - \alpha e_1 \in G_t \},$$

i.e. $L_t(z)$ is the maximum amount of cash $\alpha$ we may obtain when we change $z = (z - \alpha e_1) + \alpha e_1$ into $\alpha e_1$.

Similarly, if we define $C_t(z) = -L_t(-z)$, we obtain that

$$C_t(z) = \inf \{ \alpha \in \mathbb{R} : \alpha e_1 - z \in G_t \} = \min \{ \alpha \in \mathbb{R} : \alpha e_1 - z \in G_t \},$$

i.e. $C_t(z)$ is the minimum cost $\alpha$ expressed in cash we need to buy the financial position $z \in \mathbb{R}^d$. Indeed, we write $\alpha e_1 = z + (\alpha e_1 - z)$. 

Naturally, $C_t(z) = C_t(S_t, z)$ depends on the available quantities and prices for the risky assets, described by an exogenous vector-valued $\mathcal{F}_t$-measurable random variable $S_t$ of $\mathbb{R}_+^m$, $m \geq d$, and on the quantities $z \in \mathbb{R}^d$ to be traded.

We generally suppose that $m \geq d$ as an asset may be described by several prices and quantities offered by the market, e.g. bid and ask prices, or several pair of bid and ask prices of an order book and the associated quantities offered by the market.
The self-financing condition is:

$$\Delta V_t \in -G_t, \ t = 0, \cdots, T$$

i.e. $V_{t-1} = V_t + (-\Delta V_t)$ is such that $L_t(-\Delta V_t) \geq 0$ so that we may cancel the position $-\Delta V_t$ and change $V_{t-1}$ into $V_t$ for free.
The super-hedging problem with proportional transaction costs

Some no-arbitrage conditions are introduced for physical self-financing portfolio processes in the spirit of NA, e.g. the robust $NA^r$ condition, see the Kabanov model, and we have:

**Theorem**

$NA^r$ holds if and only if there exists strictly consistent price systems (SCPS), i.e. martingales $Z$ of $\mathbb{R}^d$ such that, for all $t = 0, \cdots, d$, $Z_t \in G_t^* = \{y \in \mathbb{R}^d : xy \geq 0, \forall x \in G_t\}$.

Then, there exists a minimal price in cash for the European claim $\xi_T \in \mathbb{R}^d$ given by

$$\sup_{Z \in SCPS, Z_0e_1 = 1} EZ_T\xi_T.$$
The super-hedging problem for general transaction costs

In practice the transaction costs are not necessary linear, see the case of order books or fixed costs.

The model is not linear so that we cannot expect dual elements characterizing a no-arbitrage condition that allow us to dually characterize the super-hedging prices.

Moreover, the no-arbitrage conditions we can imagine seem to be rather artificial, see the case of the Kabanov model where several distinct no-arbitrage exist and are difficult to compare.
A new approach based on random set

A random set is a set-valued process \((G_t)_{t=0,\ldots,T}\) such that \(G_t\) is a mapping defined on \(\Omega\) with values \(G_t(\omega)\) which are subsets of \(\mathbb{R}^d\), \(d \geq 1\).

**Definition**

We say that the set-valued process \((G_t)_{t=0,\ldots,T}\) is graph-measurable with respect to the filtration \((\mathcal{F}_t)_{t=0,\ldots,T}\) is, for all \(t \leq T\), we have:

\[
\text{graph}(G_t) := \{(\omega, x) : x \in G_t(\omega)\} \in \mathcal{F}_t \times \mathcal{B}(\mathbb{R}^d).
\]
Conditioning the random sets : conditional core

In mathematical finance, we meet some problems such as:

Find the set of all $\mathcal{F}_{t-1}$-measurable random variables $V_{t-1}$ such that $V_{t-1} \in \Gamma_t$ a.s. where $\Gamma_t$ is an $\mathcal{F}_t$-graph-measurable random set. We denote this family by $L^0(\Gamma_t, \mathcal{F}_{t-1})$.

Example 1: When $d = 1$, if $\Gamma_t(\omega) = [V_t(\omega), \infty)$ where $V_t$ is $\mathcal{F}_t$-measurable, then $V_{t-1} \in L^0(\Gamma_t, \mathcal{F}_{t-1})$ iff $V_{t-1} \geq V_t$ a.s..

We get that $L^0(\Gamma_t, \mathcal{F}_{t-1}) = L^0(\mathcal{m}(\Gamma_t|\mathcal{F}_{t-1}), \mathcal{F}_{t-1})$ where

$$\mathcal{m}(\Gamma_t|\mathcal{F}_{t-1}) = [\text{ess sup}_{\mathcal{F}_{t-1}}(V_t), \infty).$$

Example 2: When $d = 1$, if $\Gamma_t(\omega) = (-\infty, V_t(\omega)]$ where $V_t$ is $\mathcal{F}_t$-measurable, then $V_{t-1} \in L^0(\Gamma_t, \mathcal{F}_{t-1})$ iff $V_{t-1} \leq V_t$ a.s..

We get that $L^0(\Gamma_t, \mathcal{F}_{t-1}) = L^0(\mathcal{m}(\Gamma_t|\mathcal{F}_{t-1}), \mathcal{F}_{t-1})$ where

$$\mathcal{m}(\Gamma_t|\mathcal{F}_{t-1}) = (-\infty, \text{ess inf}_{\mathcal{F}_{t-1}}(V_t)].$$
Conditioning the random sets: conditional core

Theorem

Suppose that $\Gamma_t$ is a $\mathcal{F}_t$-graph-measurable random set which is a.s. closed. Then, there exists a largest $\mathcal{F}_{t-1}$-graph-measurable random set, denoted by $m(\Gamma_t|\mathcal{F}_{t-1})$ and called $\mathcal{F}_{t-1}$-conditional core of $\Gamma_t$, such that $m(\Gamma_t|\mathcal{F}_{t-1}) \subseteq \Gamma_t$ a.s.. Moreover, we have $L^0(\Gamma_t,\mathcal{F}_{t-1}) = L^0(m(\Gamma_t|\mathcal{F}_{t-1}),\mathcal{F}_{t-1})$. 
Conditioning the random sets: conditional closure and interior

The conditional closure concept arises from the following problem:

Let \( h_{t-1}(\omega, x) \) be a random function such that
\[
\omega \in \Omega \mapsto h_{t-1}(\omega, x) \text{ is } \mathcal{F}_{t-1}\text{-measurable and } \\
x \in \mathbb{R}^d \mapsto h_{t-1}(\omega, x) \in \mathbb{R} \text{ is l.s.c. a.s.}(\omega).
\]

With a complete \( \sigma \)-algebra, we may say that \( h_{t-1} \) is a normal integrand. Then, compute

\[
\text{ess sup}_{\mathcal{F}_{t-1}}\{ h_{t-1}(\omega, S_t(\omega)) : S_t \in L^0(\Gamma_t, \mathcal{F}_t) \}
\]

where \( \Gamma_t \) is a \( \mathcal{F}_t \)-graph-measurable set-valued mapping.
Conditioning the random sets: conditional closure and interior

**Theorem**

Let $\Gamma_t$ be a $\mathcal{F}_t$-graph-measurable set-valued mapping. There exists a largest $\mathcal{F}_{t-1}$-graph-measurable set-valued mapping denoted by $\mathcal{O}(\Gamma_t|\mathcal{F}_{t-1})$ and called conditional interior, such that $\mathcal{O}(\Gamma_t|\mathcal{F}_{t-1})$ is a.s. open and $\mathcal{O}(\Gamma_t|\mathcal{F}_{t-1}) \subseteq \Gamma_t$ a.s..

**Corollary**

Let $\Gamma_t$ be a $\mathcal{F}_t$-graph-measurable set-valued mapping. There exists a smallest $\mathcal{F}_{t-1}$-graph-measurable set-valued mapping denoted by $\mathcal{cl}(\Gamma_t|\mathcal{F}_{t-1})$ and called conditional closure, such that $\mathcal{cl}(\Gamma_t|\mathcal{F}_{t-1})$ is a.s. closed and $\Gamma_t \subseteq \mathcal{cl}(\Gamma_t|\mathcal{F}_{t-1})$ a.s.
Application in random optimization

**Theorem**

Suppose that the random function $h_{t-1}(\omega, x)$ is a $\mathcal{F}_{t-1}$-normal integrand. Let $\Gamma_t$ be a $\mathcal{F}_t$-graph-measurable set-valued mapping. Then,

$$\text{ess sup}_{\mathcal{F}_{t-1}} \{ h_{t-1}(\omega, S_t(\omega)) : S_t \in L^0(\Gamma_t, \mathcal{F}_t) \} = \sup_{z \in \text{cl}(\Gamma_t|\mathcal{F}_{t-1})} h_{t-1}(\omega, z).$$
Application in random optimization

**Theorem**

Suppose that the random function \( h_{t-1}(\omega, x) \) is a \( \mathcal{F}_{t-1} \)-normal integrand. Let \( S_t \) be a \( \mathcal{F}_t \)-measurable random variable. Then,

\[
\text{ess sup}_{\mathcal{F}_{t-1}} h_{t-1}(\omega, S_t(\omega)) = \sup_{z \in \text{supp}(S_t | \mathcal{F}_{t-1})} h_{t-1}(\omega, z).
\]

where \( \text{supp}(S_t | \mathcal{F}_{t-1}) \) is the conditional support of \( S_t \) knowing \( \mathcal{F}_{t-1} \).
Thank you for your attention!

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