On convergence of stochastic Mayer problems with transaction cost

Yuri Kabanov

Laboratoire de Mathématiques, Université de Franche-Comté

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There are $d$ assets which we prefer to interpret as currencies. Their quotes are given in units of a certain *numéraire* which may not be a traded security. At time $t$ the quotes are expressed by the vector of prices $S_t = (S^1_t, \ldots, S^d_t)$; its components are strictly positive. We assume that $S_0 = 1 = (1, \ldots, 1)$.

The agent’s positions can be described either by the vector of “physical” quantities $\hat{V}_t = (\hat{V}^1_t, \ldots, \hat{V}^d_t)$ or by the vector $V = (V^1_t, \ldots, V^d_t)$ of values invested in each asset; they are related as follows:

$$\hat{V}^i_t = V^i_t / S^i_t, \quad i \leq d.$$
Basic model in discrete time
Dynamics

The portfolio evolution can be described by the initial condition
\( V_{-0} = v \) (the endowments of the agent when entering the market) and the increments at dates \( t \geq 0 \):

\[
\Delta V_t^i = \hat{V}_{t-1}^i \Delta S_t^i + \Delta B_t^i,
\]

\[
B_t^i := \sum_{j=1}^{d} L_t^{ji} - \sum_{j=1}^{d} (1 + \lambda_t^{ij}) L_t^{ij},
\]

where \( L_t^{ji} \in L^0(\mathbb{R}_+, \mathcal{F}_t) \) represents the accumulated net amount transferred from the position \( j \) to the position \( i \) at the date \( t \); \( (\Delta L_t^{ij}) \), interpreted as an “order” matrix, is a control; \( (\lambda_t^{ij}) \) is the matrix of transaction costs coefficients: \( \lambda_t^{ij} \in L^0(\mathbb{R}_+, \mathcal{F}_t), \lambda_t^{ii} = 0. \)
Dynamics in terms of numéraire

The portfolio dynamics can be described by a controlled linear difference equation:

\[ \Delta V^i_t = V^i_{t-1} \Delta Y^i_t + \Delta B^i_t, \quad i = 1, \ldots, d, \]

where \( Y^i \), a “stochastic logarithm” of \( S^i \), is given by follows:

\[ \Delta Y^i_t = \frac{\Delta S^i_t}{S^i_{t-1}}, \quad Y^i_0 = 1. \]

We can take \( \Delta B_t \) as the control. Any \( \Delta L_t \in L^0(\mathcal{M}_+, \mathcal{F}_t) \) defines \( \Delta B_t \in L^0(-K_t, \mathcal{F}_t) \) where \( K_t \) is the solvency cone

\[ K_t := \left\{ x \in \mathbb{R}^d : \exists a \in \mathcal{M}_+^d \text{ such that } x^i \geq \sum_j [(1 + \lambda_{ij}^t)a^{ij} - a^{ji}] \right\}. \]

A measurable selection arguments show that any increment \( \Delta B_t \in L^0(-K_t, \mathcal{F}_t) \) is generated by a certain (in general, not unique) order \( \Delta L_t \in L^0(\mathcal{M}_+, \mathcal{F}_t) \).
Dynamics in physical units and the Cauchy formula

- The portfolio dynamics in physical units is surprisingly simple and, financially, obvious:

\[ \Delta \hat{V}_t^i = \frac{\Delta B_t^i}{S_t^i}, \quad i = 1, \ldots, d. \]

- We can write this as:

\[ \Delta \hat{V}_t = \Delta \hat{B}_t, \quad -\Delta \hat{B}_t \in \hat{K}_t := \varphi_t M_t. \]

- It follows that

\[ V_t^i = S_t^i \hat{V}_t^i = S_t^i \left( v^i + \sum_{s=0}^{t} \frac{\Delta B_s^i}{S_s^i} \right). \]

This is just the Cauchy formula for the solution of the non-homogeneous linear difference equation.
We are given
- a closed proper convex cone $K \subset \mathbb{R}^d$ such that $\text{int} \ K \supset \mathbb{R}^d_+ \setminus \{0\}$,
- a probability measure $\mu$ on the space $C^d$ of continuous functions $x$ on $[0, T]$ with values in $]0, \infty[^d$ and such that $x_0 = 1 = (1, \ldots, 1)$ and $x^1_t \equiv 1$,
- a function $U : K \times C^d_{++}[0, T] \to \mathbb{R}^+$ such that for each $f$ the function $v \mapsto U(v, f)$ is concave and increasing with respect to the componentwise partial ordering in $\mathbb{R}^d$ (i.e., induced by the cone $\mathbb{R}^d_+$).

In our terminology model $\mathbf{M}(\mu)$ is a stochastic basis $\mathbb{B} = (\Omega, \mathcal{F}, \mathbf{F} = (\mathcal{F}_t)_{t \in [0, T]}, \mathbb{P})$ satisfying the usual conditions on which is defined a process $S$ having the law $\mu$ and such that its natural filtration $\mathbf{F}^S = \mathbf{F}$. 
Fix a model $M = M(\mu)$.

We associate with $K$ and $S$ the cone-valued processes $\hat{K}_t := K/S_t$ and $\hat{K}_t^*$. In a formal way $\hat{K}_t := \varphi_t K$, where $\varphi_t : (x^1, \ldots, x^d) \mapsto (x^1/S^1_t, \ldots, x^d/S^d_t)$. Then $\hat{K}_t^* = \varphi^{-1}K^*$.

Let $B = (B_t)_{t \leq T}$ be an $\mathbb{R}^d$-valued adapted càdlàg process of bounded variation, $\text{Var} B = \text{Var}_t B$ be the sum of $\text{Var} B^i$, and $\dot{B}_t := dB_t/d\text{Var}_t B$ is an optional process. If $\dot{B}_t \in -K$ the process $B$ is called control or strategy.

For a control $B$ and $x \in K$ the processes $\hat{V} = \hat{V}^{x,B}$ has the components $\hat{V}^i := x^i + (1/S^i) \cdot B^i$ and $V = \varphi^{-1}\hat{V}$ with $V^i = S^i \hat{V}^i = S^i(x^i + (1/S^i) \cdot B^i)$.

If $S$ is a semimartingale, the product formula implies that $V^i = x^i + V^i \cdot L^i + B^i$ where $L^i := 1 + (1/S^i) \cdot S^i$. Alternatively, $V^i = x^i + \hat{V}^i \cdot S^i + B^i$.

The convex set $A(x) = A(x, M)$ of admissible strategies is formed by the controls $B$ such that $\hat{V}^{x,B} \in \hat{K}$, i.e., $\hat{V}^{x,B}_t \in \hat{K}_t$, $t \in [0, T]$. Clearly, $A(y) \supseteq A(x)$ if $y - x \in K$, $A(\lambda x) = \lambda A(x)$ $\forall \lambda > 0$. 
The Mayer problem and its stability

The aim of the control is to minimize over $A(x, M)$ the expected utility $E[U(\hat{V}^x_T, B, S)]$, i.e. to find the Bellman function for the model $M = M(\mu)$

$$u(x, M) := \sup_{B \in A(x, M)} E[U(\hat{V}^x_T, B, S)].$$ (1)

Let $M^n := M(\mu^n)$ where $\mu^n$ to $\mu$ weakly.

The question is whether $u(x, M^n)$ converges to $u(x, M)$?
**Assumption A.1.** There are two continuous functions \( m_i : [0, 1] \to \mathbb{R}_+ \) with \( m_i(0) = 0, \ i = 1, 2 \), and an integrable random variable \( \zeta \geq 0 \) such that for all \( x \in K \) and \( \alpha > 0 \)

\[
U((1 - \alpha)x, S) \geq (1 - m_1(\alpha))U(x, S) - m_2(\alpha)\zeta.
\] (2)

**Lemma**

*Suppose that A.1 holds. Then \( u \) is continuous on \( \text{int} \ K \).*
Approximations of strategies

Lemma

Let $\varepsilon \in ]0, 1]$ be such that $O_{\varepsilon}(x) \subset \text{int} \ K$. Let $B \in \mathcal{A}(x)$ and let $B^m$ be the strategy defined as follows:

$$B^m := b_0 I_{[t_0, t_1]} + \sum_{k=1}^{m-1} (B_{t_k} - B_{t_{k-1}}) I_{\Delta_k} + B_T I_{\{T\}}, \quad \Delta_k := [t_k, t_{k+1}], \quad t_k = kT/m.$$ 

Then

(i) $|\hat{V}_T^{x, B^m} - \hat{V}_T^{x, B}| \to 0$ as $m \to \infty$;

(ii) there is an $\mathbb{R}_+^d$-valued adapted càdlàg process $\xi^m = \xi^m(x, S, B)$ with jumps only at the points $t_k$ and such that $\|\xi^m\| \to 0$ as $m \to \infty$ and

$$\hat{V}_t^{x, B^m} + \xi_t^m \in \hat{K}_t \quad \text{for all } t \in [0, T].$$
Lemma

Let $\delta > 0$, $\zeta$ be an $\mathbb{R}^k$-valued r.v., and $\eta$ be a $\sigma\{\zeta\}$-measurable r.v. with values in a closed convex cone $G$. Then there is a bounded continuous function $f : \mathbb{R}^k \to G$ such that $E[|f(\zeta) - \eta| \wedge 1] < \delta$.

Lemma

Let $\delta > 0$ and let $\eta$ be a $G$-valued $F_t$-measurable r.v. Then there are \( r_i \in [0, t], \ i = 1, \ldots, M \), and a bounded continuous function $f : (\mathbb{R}^p)^M \to G$ such that $E[|f(Y_{r_1}, \ldots, Y_{r_M}) - \eta| \wedge 1] < \delta$.

Lemma

Let $\mathcal{B}_m(F)$ be the set of $F$-adapted processes constant on each $[t_k, t_{k+1}]$ with $\Delta B^m_{t_k} \in L^0(-K, F_{t_k})$. Let $\mathcal{B}_m^c(F)$ be its subset consisting of the processes such that $\Delta B^m_{t_k}$ are bounded continuous functions of values of the process $Y$ at a finite subset of $[0, t_k]$. Then for any $B^m \in \mathcal{B}_m(F)$ there are $B^m, l \in \mathcal{B}_m^c(F)$ with $\|B^m, l - B^m\| \to 0$ a.s. as $l \to \infty$. 
Assumption A.2. The value function of the problem does not depend on the model: \( u(x, M^n) =: u^n(x), \) \( u(x, M) =: u(x). \)

Proposition

Suppose that A.1 and A.2 hold. Let \( x \in \text{int} \, K. \) Then

\[
u(x) \leq \liminf_n u^n(x).
\]
**Assumption A.3.** There exist a cone $K'$ and cones $K'^n$, satisfying the following property: $K' \subset \text{int} K' \cup \{0\}$ and $K' \subset \text{int}K' \cup \{0\}$. There also exist probability measures $Q \sim P$, $Q^n \sim P^n$, $n \in \mathbb{N}$ and local $Q^n$-martingales $M^*_n$ with a local $Q$-martingale $M$ such that

$$|M_t - S_t| \in K'^*, \; P\text{-a.s.} \quad \forall t \in [0, T], \quad (3)$$

$$|M^n_t - S^n_t| \in K'^*_n, \; P^n\text{-a.s.} \quad \forall n \in \mathbb{N} \quad \forall t \in [0, T]. \quad (4)$$

**Assumption A.4.** The sequence of probability measures $P^n$ is contiguous with respect to the sequence $Q^n$, where $Q^n$ is taken form the previous assumption.

**Theorem**

*Under A.1-A.4* $u^n(x) \to u(x).$