On convergence of stochastic Mayer problems with transaction cost

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August, 2022

Basic model in discrete time

Values versus physical units

There are d assets which we prefer to interpret as currencies. Their quotes are given in units of a certain *numéraire* which may not be a traded security. At time t the quotes are expressed by the vector of prices $S_t = (S_t^1, \dots, S_t^d)$; its components are strictly positive. We assume that $S_0 = \mathbf{1} = (1, ..., 1)$.

The agent's positions can be described either by the vector of "physical" quantities $\widehat{V}_t = (\widehat{V}_t^1, \dots, \widehat{V}_t^d)$ or by the vector $V = (V_t^1, \dots, V_t^d)$ of values invested in each asset ; they are related as follows :

$$\widehat{V}_t^i = V_t^i / S_t^i, \quad i \le d.$$

Formally, $\widehat{V}_t = \varphi_t V_t$, where

$$\varphi_t : (x^1, ..., x^d) \mapsto (x^1/S_t^1, ..., x^d/S_t^d).$$

The portfolio evolution can be described by the initial condition $V_{-0} = v$ (the endowments of the agent when entering the market) and the increments at dates $t \geq 0$:

$$\Delta V_t^i = \widehat{V}_{t-1}^i \Delta S_t^i + \Delta B_t^i,$$

$$B_t^i := \sum_{j=1}^{d} L_t^{ji} - \sum_{j=1}^{d} (1 + \lambda_t^{ij}) L_t^{ij},$$

where $L_t^{ji} \in L^0(\mathbf{R}_+, \mathcal{F}_t)$ represents the accumulated net amount transferred from the position j to the position i at the date t ; (ΔL_t^{ij}), interpreted as an "order" matrix, is a control ; (λ_t^{ij}) is the matrix of transaction costs coefficients : $\lambda_t^{ij} \in L^0(\mathbf{R}_+, \mathcal{F}_t)$, $\lambda^{ii} = 0$.

Dynamics in terms of numéraire

The portfolio dynamics can be described by a controlled linear difference equation :

$$\Delta V_t^i = V_{t-1}^i \Delta Y_t^i + \Delta B_t^i, \qquad i = 1, ..., d,$$

where Y^i, a "stochastic logarithm" of S^i, is given by follows :

$$\Delta Y_t^i = \frac{\Delta S_t^i}{S_{t-1}^i}, \quad Y_0^i = 1.$$

We can take ΔB_t as the control. Any $\Delta L_t \in L^0(\mathbf{M}_+^d, \mathcal{F}_t)$ defines $\Delta B_t \in L^0(-K_t, \mathcal{F}_t)$ where K_t is the solvency cone

$$K_t := \Big\{ x \in \mathbf{R}^d : \; \exists\, a \in \mathbf{M}_+^d \text{ such that } x^i \geq \sum_j [(1+\lambda_t^{ij})a^{ij} - a^{ji}] \Big\}.$$

A measurable selection arguments show that any increment $\Delta B_t \in L^0(-K_t, \mathcal{F}_t)$ is generated by a certain (in general, not unique) order $\Delta L_t \in L^0(\mathbf{M}_+^d, \mathcal{F}_t)$.

Dynamics in physical units and the Cauchy formula

- The portfolio dynamics in physical units is surprisingly simple and, financially, obvious :

$$\Delta \widehat{V}_t^i = \frac{\Delta B_t^i}{S_t^i}, \qquad i = 1, ..., d.$$

- We can write this as :

$$\Delta \widehat{V}_t = \widehat{\Delta B}_t, \qquad -\widehat{\Delta B}_t \in \widehat{K}_t := \varphi_t M_t.$$

- It follows that

$$V_t^i = S_t^i \widehat{V}_t^i = S_t^i \left(v^i + \sum_{s=0}^{t} \frac{\Delta B_s^i}{S_s^i} \right).$$

This is just the Cauchy formula for the solution of the non-homogeneous linear difference equation.

Problem formulation

We are given

– a closed proper convex cone $K \subset \mathbb{R}^d$ such that $\operatorname{int} K \supset \mathbb{R}^d_+ \setminus \{0\}$,

– a probability measure μ on the space C^d of continuous functions x on $[0, T]$ with values in $]0, \infty[^d$ and such that $x_0 = \mathbf{1} = (1, \dots, 1)$ and $x^1_t \equiv 1$,

– a function $U : K \times C^d_{++}[0, T] \to \mathbb{R}_+$ such that for each f the function $v \mapsto U(v, f)$ is concave and increasing with respect to the componentwise partial ordering in $\mathbb{R}^d$ (i.e., induced by the cone $\mathbb{R}^d_+$).

In our terminology *model* $\mathbf{M}(\mu)$ is a stochastic basis $\mathbb{B} = (\Omega, \mathcal{F}, \mathbf{F} = (\mathcal{F}_t)_{t\in[0,T]}, \mathbf{P})$ satisfying the usual conditions on which is defined a process S having the law μ and such that its natural filtration $\mathbf{F}^S = \mathbf{F}$.

Fix a model $\mathbf{M} = \mathbf{M}(\mu)$.
We associate with K and S the cone-valued processes $\widehat{K}_t := K/S_t$ and $\widehat{K}^*_t$. In a formal way $\widehat{K}_t := \varphi_t K$, where $\varphi_t : (x^1, ..., x^d) \mapsto (x^1/S^1_t, \dots, x^d/S^d_t)$. Then $\widehat{K}^*_t = \varphi^{-1} K^*$.

Let $B = (B_t)_{t \le T}$ be an $\mathbb{R}^d$-valued adapted càdlàg process of bounded variation, $\mathrm{Var}\, B = \mathrm{Var}_t B$ be the sum of $\mathrm{Var}\, B^i$, and $\dot{B}_t := dB_t/d\mathrm{Var}_t B$ is an optional process. If $\dot{B}_t \in -K$ the process B is called *control* or *strategy*.

For a control B and $x \in K$ the processes $\widehat{V} = \widehat{V}^{x,B}$ has the components $\widehat{V}^i := x^i + (1/S^i) \cdot B^i$ and $V = \varphi^{-1} \widehat{V}$ with $V^i = S^i \widehat{V}^i = S^i(x^i + (1/S^i) \cdot B^i)$.
If S is a semimartingale, the product formula implies that $V^i = x^i + V^i \cdot L^i + B^i$ where $L^i := 1 + (1/S^i) \cdot S^i$. Alternatively, $V^i = x^i + \widehat{V}^i \cdot S^i + B^i$.
The convex set $\mathcal{A}(x) = \mathcal{A}(x, \mathbf{M})$ of *admissible strategies* is formed by the controls B such that $\widehat{V}^{x,B} \in \widehat{K}$, i.e., $\widehat{V}^{x,B}_t \in \widehat{K}_t$, $t \in [0, T]$.
Clearly, $\mathcal{A}(y) \supseteq A(x)$ if $y - x \in K$, $\mathcal{A}(\lambda x) = \lambda \mathcal{A}(x)\ \forall\, \lambda > 0$.

The Mayer problem and its stability

The aim of the control is to minimize over $\mathcal{A}(x,\mathbf{M})$ the expected utility $\mathbf{E}[U(\widehat{V}_T^{x,B},S)]$, i.e. to find the Bellman function for the model $\mathbf{M}=\mathbf{M}(\mu)$

$$u(x,\mathbf{M}) := \sup_{B\in\mathcal{A}(x,\mathbf{M})} \mathbf{E}[U(\widehat{V}_T^{x,B},S)]. \tag{1}$$

Let $\mathbf{M}^n := \mathbf{M}(\mu^n)$ where μ^n to μ weakly.

The question is whether $u(x,\mathbf{M}^n)$ converges to $u(x,\mathbf{M})$?

Continuity

Assumption A.1. There are two continuous functions $m_i : [0,1] \to \mathbb{R}_+$ with $m_i(0) = 0$, $i = 1, 2$, and an integrable random variable $\zeta \geq 0$ such that for all $x \in K$ and $\alpha > 0$

$$U((1-\alpha)x, S) \geq (1 - m_1(\alpha))U(x, S) - m_2(\alpha)\zeta. \qquad (2)$$

Lemma

Suppose that **A.1** *holds. Then u is continuous on* $\operatorname{int} K$.

Approximations of strategies

Lemma

Let $\varepsilon \in]0,1]$ be such that $\mathcal{O}_\varepsilon(x) \subset \operatorname{int} K$. Let $B \in \mathcal{A}(x)$ and let B^m be the strategy defined as follows :

$$B^m := b_0 I_{[t_0,t_1[} + \sum_{k=1}^{m-1} (B_{t_k} - B_{t_{k-1}}) I_{\Delta_k} + B_T I_{\{T\}}, \quad \Delta_k := [t_k, t_{k+1}[, \quad t_k =$$

Then
(i) $|\widehat{V}_T^{x,B^m} - \widehat{V}_T^{x,B}| \to 0$ *as* $m \to \infty$ *;*
(ii) there is an $\mathbb{R}^d_+$*-valued adapted càdlàg process* $\xi^m = \xi^m(x,S,B)$ *with jumps only at the points* t_k *and such that* $||\xi^m|| \to 0$ *as* $m \to \infty$ *and*

$$\widehat{V}_t^{x,B^m} + \xi_t^m \in \widehat{K}_t \quad \text{for all } t \in [0,T].$$

Lemma

Let $\delta > 0$, $l\zeta$ be an $\mathbb{R}^k$-valued r.v., and η be a $\sigma\{\zeta\}$-measurable r.v. with values in a closed convex cone G. Then there is a bounded continuous function $f : \mathbb{R}^k \to G$ such that $\mathbf{E}[|f(\zeta) - \eta| \wedge 1] < \delta$.

Lemma

Let $\delta > 0$ and let η be a G-valued $\mathcal{F}_t$-measurable r.v. . Then there are $r_i \in [0, t]$, $i = 1, ..., M$, and a bounded continuous function $f : (\mathbb{R}^p)^M \to G$ such that $\mathbf{E}[|f(Y_{r_1}, ..., Y_{r_M}) - \eta| \wedge 1] < \delta$.

Lemma

Let $\mathscr{B}_m(\mathbf{F})$ be the set of $\mathbf{F}$-adapted processes constant on each $[t_k, t_{k+1}[$ with $\Delta B^m_{t_k} \in L^0(-K, \mathcal{F}_{t_k})$. Let $\mathscr{B}^c_m(\mathbf{F})$ be its subset consisting of the processes such that $\Delta B^m_{t_k}$ are bounded continuous functions of values of the process Y at a finite subset of $[0, t_k]$. Then for any $B^m \in \mathscr{B}_m(\mathbf{F})$ there are $B^{m,l} \in \mathscr{B}^c_m(\mathbf{F})$ with $||B^{m,l} - B^m|| \to 0$ a.s. as $l \to \infty$.

Assumption A.2. The value function of the problem does not depend on the model : $u(x, \mathbf{M}^n) =: u^n(x)$, $u(x, \mathbf{M}) =: u(x)$.

Proposition

Suppose that **A.1** *and* **A.2** *hold. Let* $x \in \operatorname{int} K$. *Then*

$$u(x) \le \liminf_n u^n(x).$$

Assumption A.3. There exist a cone K' and cones K'^n, satisfying the following property : $K' \subset \operatorname{int} K' \cup \{0\}$ and $K' \subset \operatorname{int} K' \cup \{0\}$. There also exist probability measures $\mathbf{Q} \sim \mathbf{P}$, $\mathbf{Q}^n \sim \mathbf{P}^n$, $n \in \mathbb{N}$ and local $\mathbf{Q}^n$-martingales M_t^n with a local $\mathbf{Q}$-martingale M such that

$$|M_t - S_t| \in K'^*, \quad \mathbf{P}\text{-a.s.} \quad \forall t \in [0, T], \tag{3}$$

$$|M_t^n - S_t^n| \in K_n'^*, \quad \mathbf{P}^n\text{-a.s.} \quad \forall n \in \mathbb{N} \quad \forall t \in [0, T]. \tag{4}$$

Assumption A.4. The sequence of probability measures $\mathbf{P}^n$ is contiguous with respect to the sequence $\mathbf{Q}^n$, where $\mathbf{Q}^n$ is taken form the previous assumption.

Theorem

Under **A.1-A.4** $u^n(x) \to u(x)$.

Bayraktar, E., Dolinskyi, L., Dolinsky, Y. : Extended weak convergence and utility maximisation with proportional transaction costs. Finance and Stochastics, **24**, 4, 1013–1034 (2020)