Full inference for the anisotropic fractional Brownian field.

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The anisotropic fractional Brownian field.

- Non-stationary Gaussian random field Z, with stationary increments.
- Characterized by a semi-variogram of the form

$$
\begin{aligned}
v_0(h;\tau,\beta) &= \frac{1}{2}\mathbb{E}\left((Z(x+h)-Z(x))^2\right), \\
&= \frac{1}{2}\int_{\mathbb{R}^2}\left|1-e^{i\langle h,w\rangle}\right|^2 \tau(w)|w|^{-2\beta(w)-d}dw.
\end{aligned}
$$

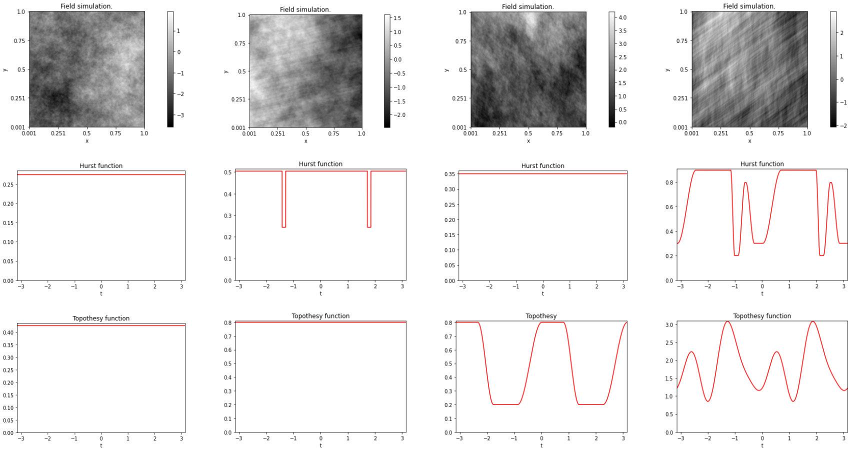
- τ and β are two non-negative homogeneous functions, called the the topothesy and Hurst functions.

$$
\tau(w)=\tau\left(\frac{w}{|w|}\right) \text{ and } \beta(w)=\beta\left(\frac{w}{|w|}\right).
$$

- Issue : estimate these functional parameters of the field.

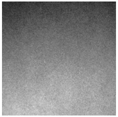
[A. Bonami & A. Estrade, J Fourier Anal Appl, 2003]

A model for image micro-textures.

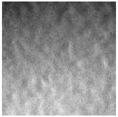


Texture classification.

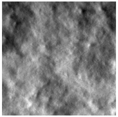
Microscopic images of photographic films (source : Paul Messier, MoMA, NY).



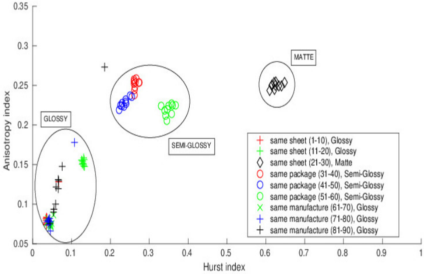
Glossy



Semi-glossy



Matte



Hurst index : $H = \arg\min_s \{\beta(s), \tau(s) > 0\}$.

Asymptotic topothesy : $\tau^*(s) = \tau(s)$ if $\beta(s) = H$ and 0 otherwise.

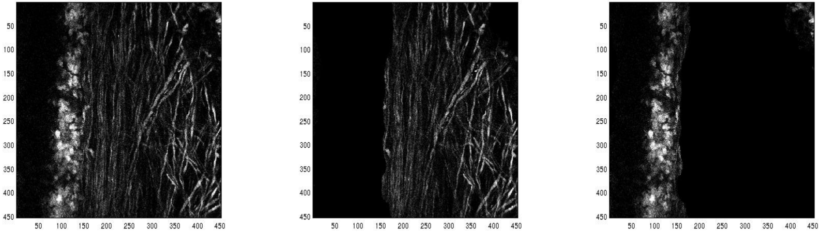
Anisotropy index : $I = \frac{\sqrt{\int (\tau^*(s) - \overline{\tau^*})^2 ds}}{\overline{\tau^*}}$, with $\overline{\tau^*} = \int \tau^*(s) ds$.

[FR, Stat & Comput, 2018 ; Spatial Stat, 2017].

Texture segmentation.

Mouse biphoton microscopy (source : F. Debarbieux, La Timone, Marseille) :

Localisation of neurons (anisotropic patterns) and inflammatory cells (isotropic patterns).



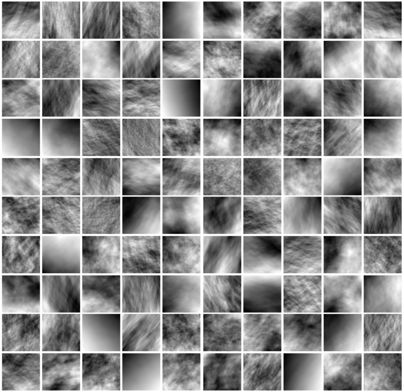
Framework of anisotropic multifractional Brownian field :

(Extension of the mfbf of [Benassi, Jaffard, Roux, 1997 ; Peltier & Vehel, 1996])

$$Z(x) = \int_{\mathbb{R}^2} \left(e^{i\langle x, w \rangle} - 1 \right) \sqrt{\tau_x(w)} |x|^{-\beta_x(w) - 1} d\widehat{W}(w)$$

[FR, Stat Sinica, 2016 ; H. Vu and FR, Stoch Process Appl, 2020.]

Texture simulation.



PyAFBF (https ://fjprichard.github.io/PyAFBF/), [FR, JOSS, 2022].

- A Python library for sampling image textures from the anisotropic fractional Brownian field.
- Motivation : Infer models from examples to simulate realistic textures.

Estimation of the Hurst function.

Let Z be an AFBF of hurst function β.

- Window Radon transforms $R(Z)(\theta)$ of Z parallel to a direction θ is a fractional Brownian motion of index $\beta(\theta^{\perp}) + \frac{1}{2}$.
- Estimate $\beta(\theta^{\perp})$ by inferring the Hurst index of $R(Z)(\theta)$.
- In practice, discretization issues that restrict the estimation to the horizontal and vertical directions.
- Inaccurate, especially for low values of the Hurst parameters.

[H. Biermé, FR, ESAIM PS, 2008].

Estimation of the asymptotic topothesy.

- Z^l : field observed on a grid $\left\{\left(\frac{i}{l},\frac{j}{l}\right)\in[\![1,l]\!]^2\right\}$.
- Increments $V^l_{s,\varphi}=v_{s,\varphi}*Z^l$ at scale s in direction φ.
- Quadratic variations : $W^N_{s,\varphi}=\frac{1}{N_e}\sum_m(V^N_{s,\varphi}[m])^2$.
- Breuer-Major Theorem $\rightarrow$ asymptotic anormality (as l tends to $+\infty$) :

$$\log(W^l_{s,\varphi})=H\ \log(s^2)+\log(\gamma_{H,\tau^*}(\varphi))+\epsilon^l_u,$$

where

$$\gamma_{H,\tau^*}(\varphi)=\tau^*\circledast\Gamma_H(\varphi)\,\text{with}\,\Gamma_H(\varphi)=\int_{\mathbb{R}^+}\left|\hat{v}\left(\rho\varphi\right)\right|^2\rho^{-2H-1}d\rho.$$

- An inverse problem : Minimize

$$\mathcal{J}(\tau)=\sum_{\varphi}\left(\tilde{\gamma}(\varphi)-\Gamma_{\tilde{H}}\circledast\tau(\varphi)\right)^2+\lambda|\tau|^2_W.$$

where $\lambda>0$ and $|\cdot|_W$ is a Sobolev norm.

A turning-band approach.

- Semi-variogram of an AFBF (in polar coordinates) :

$$v_0(h;\tau,\beta) = \frac{1}{2}\int_{-\pi/2}^{\pi/2} \nu_{\beta(\theta)}\tau(\theta)\,|\langle h, u(\theta)\rangle|^{2\beta(\theta)}\,d\theta,$$

with $u(\theta) = (\cos\theta, \sin\theta)$ and a constant ν_H.

- Can be approximated by a semi-variogram of the form

$$v(x;\tau,\beta) = \frac{1}{2}\sum_{m=1}^{M} \lambda_m \tilde{\tau}(\theta_m)|\langle x, u(\theta_m)\rangle|^{2\beta(\theta_m)},$$

for some appropriate angles θ_m in $[-\frac{\pi}{2}, \frac{\pi}{2}]$ and positive weights λ_m.

- Corresponds to the semi-variogram of a turning-band field

$$Z_M(x) = \sum_{m=1}^{M} \sqrt{\lambda_m \tilde{\tau}(\theta_m)}\, Y_m(\langle x, u(\theta_m)\rangle),$$

Y_m being a fractional Brownian field of Hurst index $\beta(\theta_m)$.

[H. Biermé, L. Moisan, FR, J Comput Graphic Stat, 2015].

Inference setting.

- $Y = (Y[i])_i$: image at some grid points $i \in [\![1, l]\!]^2$,
- Z : AFBF with unknown semi-variogram $v(\cdot; \tau, \beta)$,
- $W = (W[i])_i$ centered Gaussian noise of variance τ_0.
- Observation model :

$$Y[i] = Z\left(\frac{i}{l}\right) + W[i], i \in [\![1, l]\!]^2.$$

- Theoretical semi-variogram of Y :

$$w(x; \tau, \beta) = \tau_0 + v(x; \tau, \beta)$$

- Empirical semi-variogram of Y at some lags $(x_n)_n$:

$$\widehat{w}_n = \frac{1}{N_n} \sum_i (Y[i + x_n] - Y[i])^2.$$

The inverse problem

- Minimize the least-square criterion

$$h(\tau,\beta) = \frac{1}{2}\sum_{n=1}^{N}\left(w(x_n;\tau,\beta) - \widehat{w}_n\right)^2.$$

- Function representations :

$$\tau(\theta) = \sum_{j=1}^{J}\tau_j T_j(\theta) \text{ and } \beta(\theta) = \sum_{k=1}^{K}\beta_k B_k(\theta).$$

- h as a non-linear separable least square criterion

$$h(\tau,\beta) = \frac{1}{2}\sum_{n=1}^{N}(F_n(\beta)\tau - \widehat{w}_n)^2,$$

F_n being a vector-valued function with components $F_n(\beta)_{nj} = v(x_n; T_j, \beta)$ for $j \neq 0$ and $F(\beta)_{n0} = 1$.

A variable projection method.

VARPRO [Golub and Peyrera, 2003] :

- Define

$$g(\beta) = h(\tau^*(\beta), \beta),$$

where, for a fixed β, $\tau^*(\beta) \in \arg\min_\tau h(\tau, \beta)$.

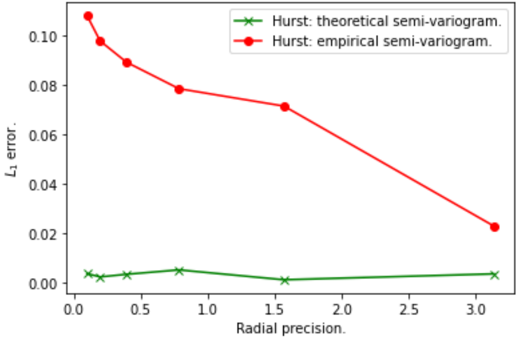
- Minimize g instead of h (with a Gauss-Newton method).

Our implementation :

- Multi-grid approach : successive minimization in embedded finite dimensional subspaces of piecewise constant functions.
- Definition of a "non-redundant" set of lags (x_n) to avoid problem to be ill-posed.
- Levenberg-Marquardt to find minimizers of h w.r.t. τ and g.

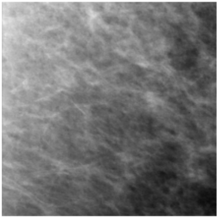
lsq_linear, **least_square** of package *Optimize* of Python library *Scipy*.
- Constraints to ensure that $\beta \in (0, 1)$ and $\tau \geq 0$.

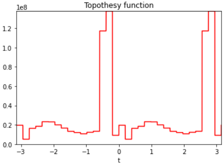
Numerical study.

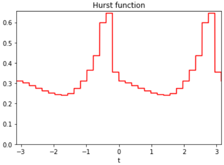


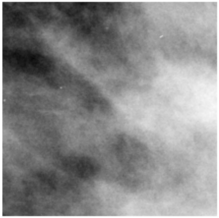
- Radial precision : maximal size of the intervals on which τ and β are piecewise constant.
- Error : mean absolute difference between the estimated and true values of the constants of β.
- Number of experiments : 100.
- Mean computational times (8 to 31 seconds).

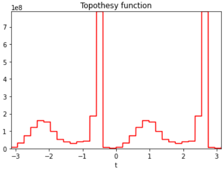
Estimating models from textures.

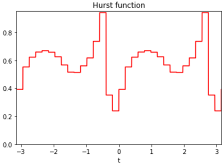










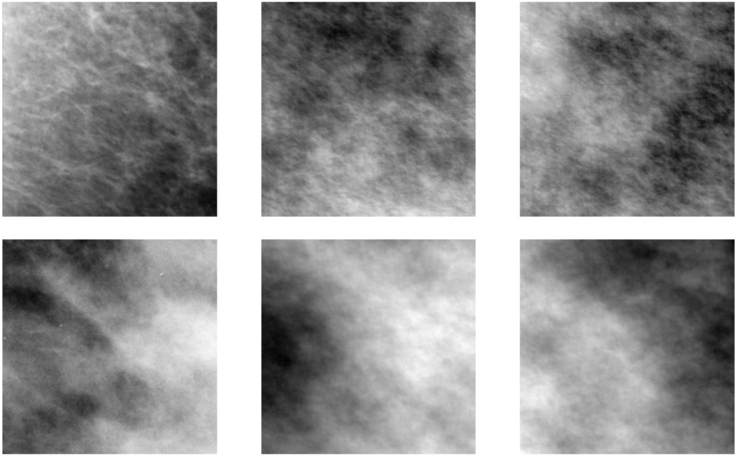


Mammogram textures

Topothesy function

Hurst function

Sampling realistic textures.



Real Synthetic 1 Synthetic 2

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F. Richard, PyAFBF : a Python library for sampling image textures from the anisotropic fractional Brownian field.. Journal of Open Source Software, 7(75), 2022.

H. Vu and F. Richard, Analysis of image texture heterogeneity using anisotropic multifractional Brownian fields, Stochastic Processes and their Applications, 130(8), 4667-4692, 2020.

F. Richard, Anisotropy of Hölder Gaussian random fields : characterization, estimation, and application to image textures, Statistics and Computing, 1–14, 2016.

F. Richard, Some anisotropy indices for the characterization of Brownian textures and their application to breast images, Spatial Statistics, 18 :147–162, 2016.

F. Richard, Tests of isotropy for rough textures of trended images, Statistica Sinica, 26 :1279-1304, 2016.

F. Richard, Analysis of anisotropic Brownian textures and application to lesion detection in mammograms, Procedia Environmental Sciences, 27 :16-20, 2015.

H. Biermé, L. Moisan and F. Richard. A turning-band method for the simulation of anisotropic fractional Brownian fields, Journal of Computational and Graphical Statistics, 24(3) :885-904, 2015.

H. Biermé and F. Richard, "Analysis of Texture Anisotropy Based on Some Gaussian Fields with Spectral Density", Springer Proceedings : Mathematical Image Processing, Editor M. Bergounioux, pp. 59-73, 2011.

F. Richard and H. Biermé, "Statistical Tests of Anisotropy for Fractional Brownian Textures. Application to Full-field Digital Mammography", Journal of Mathematical Imaging and Vision, 36(3) :227-240, 2010.

H. Biermé, F. Richard, M. Rachidi, C.L. Benhamou, "Anisotropic texture modeling and applications to medical image analysis", ESAIM Proceedings : Mathematical Methods for Imaging and Inverse Problems, Editor H. Ammari, vol. 26, pp. 100-122, 2009.

H. Biermé and F. Richard, "Estimation of anisotropic Gaussian fields through Radon transform", ESAIM : Probability and Statistics, 12(1) : 30–50, january, 2008.