Full inference
for the anisotropic fractional Brownian field.

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The anisotropic fractional Brownian field.

- Non-stationary Gaussian random field $Z$, with stationary increments.
- Characterized by a semi-variogram of the form

$$v_0(h; \tau, \beta) = \frac{1}{2} \mathbb{E} \left( (Z(x + h) - Z(x))^2 \right),$$

$$= \frac{1}{2} \int_{\mathbb{R}^2} \left| 1 - e^{i\langle h, w \rangle} \right|^2 \tau(w) |w|^{-2\beta(w) - d} \, dw.$$

- $\tau$ and $\beta$ are two non-negative homogeneous functions, called the topothesy and Hurst functions.

$$\tau(w) = \tau \left( \frac{w}{|w|} \right) \quad \text{and} \quad \beta(w) = \beta \left( \frac{w}{|w|} \right).$$

- Issue: estimate these functional parameters of the field.

[A. Bonami & A. Estrade, J Fourier Anal Appl, 2003]
A model for image micro-textures.
Texture classification.

Microscopic images of photographic films (source: Paul Messier, MoMA, NY).

Hurst index: \( H = \arg \min_s \{ \beta(s), \tau(s) > 0 \} \).

Asymptotic topothesy: \( \tau^*(s) = \tau(s) \) if \( \beta(s) = H \) and 0 otherwise.

Anisotropy index: \( I = \frac{\sqrt{\int (\tau^*(s) - \overline{\tau}^*)^2 ds}}{\overline{\tau}^*}, \) with \( \overline{\tau}^* = \int \tau^*(s) ds \).

[FR, Stat & Comput, 2018; Spatial Stat, 2017].
Texture segmentation.

Mouse biphoton microscopy (source: F. Debarbieux, La Timone, Marseille):
Localisation of neurons (anisotropic patterns) and inflammatory cells (isotropic patterns).

Framework of anisotropic multifractional Brownian field:
(Extension of the mfbf of [Benassi, Jaffard, Roux, 1997; Peltier & Vehel, 1996])

\[
Z(x) = \int_{\mathbb{R}^2} \left( e^{i\langle x, w \rangle} - 1 \right) \sqrt{\tau_x(w)} |x|^{-\beta_x(w)-1} d\hat{W}(w)
\]

Texture simulation.

PyAFBF (https://fjprichard.github.io/PyAFBF/), [FR, JOSS, 2022].

- A Python library for sampling image textures from the anisotropic fractional Brownian field.
- Motivation: Infer models from examples to simulate realistic textures.
Estimation of the Hurst function.

Let $Z$ be an AFBF of hurst function $\beta$.

- Window Radon transforms $R(Z)(\theta)$ of $Z$ parallel to a direction $\theta$ is a fractional Brownian motion of index $\beta(\theta^\perp) + \frac{1}{2}$.
- Estimate $\beta(\theta^\perp)$ by inferring the Hurst index of $R(Z)(\theta)$.
- In practice, discretization issues that restrict the estimation to the horizontal and vertical directions.
- Inaccurate, especially for low values of the Hurst parameters.

[H. Biermé, FR, ESAIM PS, 2008].
Estimation of the asymptotic topothesy.

- \( Z^I \): field observed on a grid \( \left\{ \left( \frac{i}{I}, \frac{j}{I} \right) \in [1, I]^2 \right\} \).

- Increments \( V_{s,\varphi}^I = \nu_{s,\varphi} \ast Z^I \) at scale \( s \) in direction \( \varphi \).

- Quadratic variations: \( W_{s,\varphi}^N = \frac{1}{N_e} \sum m(V_{s,\varphi}^N[m])^2 \).

- Breuer-Major Theorem \( \rightarrow \) asymptotic anormality (as \( I \) tends to \( +\infty \)):
  \[
  \log(W_{s,\varphi}^I) = H \log(s^2) + \log(\gamma_{H,\tau^*}(\varphi)) + \epsilon_u^I,
  \]
  where
  \[
  \gamma_{H,\tau^*}(\varphi) = \tau^* \otimes \Gamma_H(\varphi) \text{ with } \Gamma_H(\varphi) = \int_{\mathbb{R}^+} |\hat{\nu}(\rho \varphi)|^2 \rho^{-2H-1} d\rho.
  \]

- An inverse problem: Minimize
  \[
  J(\tau) = \sum_{\varphi} (\tilde{\gamma}(\varphi) - \tilde{\Gamma}_H \otimes \tau(\varphi))^2 + \lambda |\tau|_W^2.
  \]
  where \( \lambda > 0 \) and \( |\cdot|_W \) is a Sobolev norm.
A turning-band approach.

- Semi-variogram of an AFBF (in polar coordinates):
  \[ v_0(h; \tau, \beta) = \frac{1}{2} \int_{-\pi/2}^{\pi/2} \nu(\theta) \tau(\theta) |\langle h, u(\theta) \rangle|^{2\beta(\theta)} d\theta, \]
  with \( u(\theta) = (\cos \theta, \sin \theta) \) and a constant \( \nu_H \).

- Can be approximated by a semi-variogram of the form
  \[ v(x; \tau, \beta) = \frac{1}{2} \sum_{m=1}^{M} \lambda_m \tilde{\tau}(\theta_m) |\langle x, u(\theta_m) \rangle|^{2\beta(\theta_m)}, \]
  for some appropriate angles \( \theta_m \) in \([-\pi/2, \pi/2]\) and positive weights \( \lambda_m \).

- Corresponds to the semi-variogram of a turning-band field
  \[ Z_M(x) = \sum_{m=1}^{M} \sqrt{\lambda_m \tilde{\tau}(\theta_m)} Y_m(\langle x, u(\theta_m) \rangle), \]
  \( Y_m \) being a fractional Brownian field of Hurst index \( \beta(\theta_m) \).

Inference setting.

- $Y = (Y[i])_i$: image at some grid points $i \in [1, I]^2$,
- $Z$: AFBF with unknown semi-variogram $v(\cdot; \tau, \beta)$,
- $W = (W[i])_i$: centered Gaussian noise of variance $\tau_0$.
- Observation model:
  \[
  Y[i] = Z \left( \frac{i}{I} \right) + W[i], \; i \in [1, I]^2.
  \]
- Theoretical semi-variogram of $Y$:
  \[
  w(x; \tau, \beta) = \tau_0 + v(x; \tau, \beta)
  \]
- Empirical semi-variogram of $Y$ at some lags $(x_n)_n$:
  \[
  \hat{w}_n = \frac{1}{N_n} \sum_i (Y[i + x_n] - Y[i])^2.
  \]
The inverse problem

- Minimize the least-square criterion

\[ h(\tau, \beta) = \frac{1}{2} \sum_{n=1}^{N} (w(x_n; \tau, \beta) - \hat{w}_n)^2. \]

- Function representations:

\[ \tau(\theta) = \sum_{j=1}^{J} \tau_j T_j(\theta) \quad \text{and} \quad \beta(\theta) = \sum_{k=1}^{K} \beta_k B_k(\theta). \]

- \( h \) as a non-linear separable least square criterion

\[ h(\tau, \beta) = \frac{1}{2} \sum_{n=1}^{N} (F_n(\beta) \tau - \hat{w}_n)^2, \]

\( F_n \) being a vector-valued function with components

\[ F_n(\beta)_{nj} = v(x_n; T_j, \beta) \text{ for } j \neq 0 \text{ and } F(\beta)_{n0} = 1. \]
A variable projection method.

VARPRO [Golub and Peyrera, 2003] :

- Define
  
  \[ g(\beta) = h(\tau^*(\beta), \beta), \]

  where, for a fixed \( \beta \), \( \tau^*(\beta) \in \arg \min_{\tau} h(\tau, \beta) \).

- Minimize \( g \) instead of \( h \) (with a Gauss-Newton method).

Our implementation :

- Multi-grid approach : successive minimization in embedded finite dimensional subspaces of piecewise constant functions.

- Definition of a "non-redundant" set of lags \( (x_n) \) to avoid problem to be ill-posed.

- Levenberg-Marquardt to find minimizers of \( h \) w.r.t. \( \tau \) and \( g \).

  \texttt{lsq\_linear}, \texttt{least\_square} of package \textit{Optimize} of Python library \textit{Scipy}.

- Constraints to ensure that \( \beta \in (0, 1) \) and \( \tau \geq 0 \).
Numerical study.

- Radial precision: maximal size of the intervals on which $\tau$ and $\beta$ are piecewise constant.
- Error: mean absolute difference between the estimated and true values of the constants of $\beta$.
- Number of experiments: 100.
- Mean computational times (8 to 31 seconds).
Estimating models from textures.
Sampling realistic textures.

Real  

Synthetic 1  

Synthetic 2
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F. Richard, Analysis of anisotropic Brownian textures and application to lesion detection in mammograms, Procedia Environmental Sciences, 27:16-20, 2015.


